ONE-DIMENSIONAL MODEL OF THE REARRANGEMENT AND DISSOCIATION PROCESSES. PROBABILITY AMPLITUDES AND CROSS-SECTIONS

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In the present paper we present some further results for the quantum mechanical system of three collinear particles, already studied in our previous work. The expressions for the probability amplitudes and cross-sections for all possible scattering processes are given explicitly. The moduli of the amplitudes are expressed in terms of elementary functions. The reversibility and flux conservation rules are discussed on the basis of the configuration space scattering theory. The system in which one pair interacts *via* the hard-core type interaction is studied as a special case of the general model and discussed in detail.

1. *Introduction*

In paper [1] (hereafter referred to as I) we have discussed the quantum-mechanical scattering model of three particles in one dimension. In the model considered the two-body interactions were described with the aid of a boundary condition stating that the logarithmic derivative of the wave function with respect to the interparticle distance x is constant for $x = 0$, *i. e.*:

$$\frac{\Psi'(x)}{\Psi(x)} = -\alpha \quad \text{for} \quad x = 0. \tag{1}$$

Such a zero-range potential allows for the existence of a single bound state of a pair, of energy $E_b = -\alpha^2$. The three-body system which we construct with the two two-body subsystems of this kind represents therefore the simple boundary condition model (BCM). In this model exchange and dissociation processes are possible.

As it was shown in I, the solution of the Schrödinger equation for the system under consideration can be adopted from a certain acoustic diffraction problem, the solution of which was given by Malyuzhinetz [2], [3]. Employing this fact the exact expressions for

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the wave function and for the probability amplitudes for the elastic, rearrangement and dissociation processes were written in terms of certain special functions [2]–[4].

In Sections 2 and 3 of this paper we recall briefly the definitions, notations and basic concepts of I. In Section 4 we give the expressions for the probability amplitudes of all scattering processes of our model. These expressions are employed to derive the expressions for scattering cross-sections, this being done in Section 6.

It is remarkable that the cross-sections are expressed in terms of simple elementary functions. The derived expressions are then used to illustrate the reversibility and unitarity conditions. The model in which one of the two-body potentials is of the hard core type and the other one is of the type described by (1) is treated as a limiting case of the general model, this procedure being described in the end of Section 3. The cross-sections for the model with one hard core potential are given in Section 7. The behaviour of cross-sections as functions of mass ratio and energy are also studied.

In our earlier work we have constructed the Faddeev equations for the model under consideration and discussed the case of three particles with equal masses and two-body potentials of equal strength [5].

A particular case of the model solved here was studied earlier by Nussenzweig [6] and is discussed in Section 4. In his model two particles of equal masses interact with a fixed center of force.

After this paper has been written in its present form, a paper by Mc Guire and Hurst appeared [7] in which a solution for the same one-dimensional boundary condition model is presented independently. Mc Guire and Hurst did not use the solution of the acoustic problem by Malyuzhinetz [2], [3], but rather adopted a method outlined by Williams [8]. The relation between the special functions of Malyuzhinetz and the Barnes unsymmetric double gamma functions used by Williams is given in the Appendix. The closest correspondence occurs between the Section 6 of our paper and the final results of the above mentioned paper.

Our model is a one-dimensional counterpart of the so-called boundary condition model (BCM) studied in nuclear physics (see for instance [9], [10]).

2. *Description of the three-particle system*

We recall first the notation and some definitions introduced in I. For the system of three collinear particles with masses m_i and position and momentum coordinates in the center of mass system (c.m.s.) r_i and k_i respectively we define the following coordinates

$$s_3 = (r_1 - r_2)\left[\frac{2m_1 m_2}{m_1 + m_2}\right]^{\frac{1}{2}},$$

$$t_3 = \left(\frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} - r_3\right)\left[\frac{2m_3(m_1 + m_2)}{m_1 + m_2 + m_3}\right]^{\frac{1}{2}}, \tag{2}$$

$$p_3 = \frac{m_2 k_1 - m_1 k_2}{[2m_1 m_2(m_1 + m_2)]^{\frac{1}{2}}}, \quad q_3 = \frac{m_3(k_1 + k_2) - (m_1 + m_2)k_3}{[2m_3(m_1 + m_2)(m_1 + m_2 + m_3)]^{\frac{1}{2}}}, \tag{3}$$

as well as the coordinate systems (s_1, t_1) and (p_1, q_1) the definitions of which may be obtained from the Eq. (2) and (3) by cyclic interchange of the indices. In the following the notation (s, t) will denote one of the orthogonally equivalent systems (s_1, t_1) or (s_3, t_3).

We study the system of three particles described by the Schrödinger equation of the form (we put $\hbar = 1$)

$$-\Delta\Psi(s, t) = E\Psi(s, t), \tag{4}$$

$$\frac{\partial}{\partial s_3}\Psi(s, t)+\alpha_3\Psi(s, t) = 0 \quad \text{for} \quad s_3 = 0, \tag{4.1}$$

$$\frac{\partial}{\partial s_1}\Psi(s, t)+\alpha_1\Psi(s, t) = 0 \quad \text{for} \quad s_1 = 0. \tag{4.2}$$

The boundary conditions (4.1) and (4.2) describe the action of the zero-range impenetrable and "noncentral" potentials V_3 and V_1 between the pairs (1, 2) and (2, 3) respectively. The solutions for the two particle Schrödinger equation with the potential of this kind were discussed in I. In our model the two-body constants α_1 and α_3 are negative and real. We choose the ordering of particles to be 1 2 3; in this case motion of particles is confined to the region $s_1 < 0$ and $s_3 < 0$ in the (s, t) plane. In this region owing to the fact that $\alpha_1 < 0$ and $\alpha_3 < 0$ there is a possibility of formation of the bound state of the pair (1, 2) with the binding energy $-\alpha_3^2$ and of a bound state of the pair (2, 3) with the binding energy $-\alpha_1^2$. There is no such possibility for the pair (1, 3). We introduce the polar coordinates (r, φ) defined as follows:

$$\begin{aligned} s_3 &= r\sin(\Phi+\varphi), & s_1 &= -r\sin(\Phi-\varphi), \\ t_3 &= r\cos(\Phi+\varphi), & t_1 &= r\cos(\Phi-\varphi), \end{aligned} \tag{5}$$

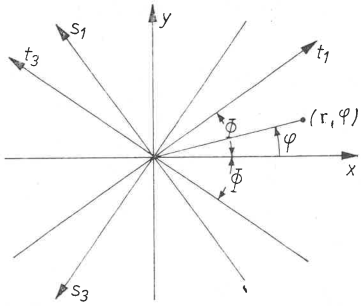

Fig. 1

where the angle Φ is related in the following way to the particle masses m_i:

$$\sin 2\Phi = \left[\frac{m_2(m_1+m_2+m_3)}{(m_1+m_2)(m_2+m_3)}\right]^{\frac{1}{2}}, \quad \cos 2\Phi = \left[\frac{m_1 m_3}{(m_1+m_2)(m_2+m_3)}\right]^{\frac{1}{2}},$$

$$\operatorname{tg} 2\Phi = \left[\frac{m_2(m_1+m_2+m_3)}{m_1 m_3}\right]^{\frac{1}{2}}. \tag{6}$$

The coordinate systems (s, t) and (r, φ) are shown in Fig. 1.

Employing the coordinates (r, φ) we may write the Schrödinger equation (4) in the following form:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}+E\right)\Psi(r,\varphi)=0, \tag{7}$$

$$\frac{1}{r}\frac{\partial}{\partial \varphi}\Psi(r;\varphi)+i\sqrt{E}\sin\Theta_-\Psi(r,\varphi)=0 \quad \text{for} \quad \varphi=-\Phi, \tag{7.1}$$

$$\frac{1}{r}\frac{\partial}{\partial \varphi}\Psi(r,\varphi)-i\sqrt{E}\sin\Theta_+\Psi(r,\varphi)=0 \quad \text{for} \quad \varphi=\Phi, \tag{7.2}$$

where the complex numbers Θ_- and Θ_+ are related in the following way to the constants α_3 and α_1 and to the total energy E:

$$\Theta_- = \text{Arctg}\frac{i\alpha_3}{\sqrt{E+\alpha_3^2}}=\frac{1}{2i}\ln\frac{\sqrt{E+\alpha_3^2}-\alpha_3}{\sqrt{E+\alpha_3^2}+\alpha_3}, \tag{8}$$

$$\Theta_+ = \text{Arctg}\frac{i\alpha_1}{\sqrt{E+\alpha_1^2}}=\frac{1}{2i}\ln\frac{\sqrt{E+\alpha_1^2}-\alpha_1}{\sqrt{E+\alpha_1^2}+\alpha_1}, \tag{9}$$

$$\sqrt{E}\cos\Theta_- = \sqrt{E+\alpha_3^2}, \qquad -i\sqrt{E}\sin\Theta_- = \alpha_3, \tag{10}$$

$$\sqrt{E}\cos\Theta_+ = \sqrt{E+\alpha_1^2}, \qquad -i\sqrt{E}\sin\Theta_+ = \alpha_1. \tag{11}$$

We choose the branch of the square root with the positive imaginary part and the logarithm and arc tangent functions are understood in the sense of their principal values. This choice is dictated by the condition of obtaining the physically admissible solution $\Psi(r, \varphi)$.

In the scattering problem we have to distinguish three energy regions for the total energy E. Assuming first that

$$-\alpha_3^2 < -\alpha_1^2, \tag{12}$$

we determine these regions as follows:

Region I:

$$-\alpha_3^2 < E < -\alpha_1^2, \quad E=-n^2<0. \tag{13}$$

In this region the only possible scattering process in the three particle system is the elastic scatering:

$$(1,2)+3 \to (1,2)+3. \tag{14}$$

We say that the only open channel is channel 3.

The argument of the Arctg function in Eq. (9) is real and Θ_+ is a real number in the interval $\left(-\frac{\pi}{2}, 0\right)$ $\left(\text{approaching } -\frac{\pi}{2}\right.$ at the upper threshold of the considered energy region$\Big)$. On the other hand, the "angle" Θ_- should be written in this energy region in the following form:

$$\Theta_- = -i\Theta_-^n - \frac{\pi}{2}, \tag{15}$$

where

$$\Theta_-^n = \frac{1}{2} \ln \frac{\alpha_3 - \sqrt{\alpha_3^2 - n^2}}{\alpha_3 + \sqrt{\alpha_3^2 - n^2}} > 0. \tag{16}$$

Region II:

$$-\alpha_1^2 < E < 0, \quad E = -l^2 < 0. \tag{17}$$

In this energy region two channels are open, namely channel 3 and channel 1 corresponding to the formation of the bound states of the pair (1, 2) and (2, 3) respectively. Thus we have two possible initial and final situations and the energetically admissible processes are elastic and exchange scatterings. The quantities Θ_+ and Θ_- may now be written in the following form:

$$\Theta_\pm = -i\Theta_\pm^l - \frac{\pi}{2}, \tag{18}$$

where

$$\Theta_-^l = \frac{1}{2} \ln \frac{\alpha_3 - \sqrt{\alpha_3^2 - l^2}}{\alpha_3 + \sqrt{\alpha_3^2 - l^2}} > 0, \tag{19}$$

$$\Theta_+^l = \frac{1}{2} \ln \frac{\alpha_1 - \sqrt{\alpha_1^2 - l^2}}{\alpha_1 + \sqrt{\alpha_1^2 - l^2}} > 0. \tag{20}$$

Region III:

$$E = k^2 > 0, \ \sqrt{E} = k > 0. \tag{21}$$

Besides channels 1 and 3, channel 0 corresponding to the free movement of three particles is also open. The break-up and recombination processes are energetically admissible. In this region we have

$$\Theta_\pm = -i\Theta_\pm^k, \tag{22}$$

where

$$\Theta_-^k = \frac{1}{2} \ln \frac{\sqrt{E + \alpha_3^2} - \alpha_3}{\sqrt{E + \alpha_3^2} + \alpha_3} > 0, \tag{23}$$

$$\Theta_+^k = \frac{1}{2} \ln \frac{\sqrt{E + \alpha_1^2} - \alpha_1}{\sqrt{E + \alpha_1^2} + \alpha_1} > 0. \tag{24}$$

Employing the definitions (5) and the relations (10), (11) we can express the unnormalized initial and final state functions for channels 3 and 1 in the following form:

$$\psi_3^i(s_3, t_3) = e^{-\alpha_3 s_3 + i\sqrt{E+\alpha^2_3}\, t_3} = e^{-i\sqrt{E}\, r\cos(\Phi+\varphi-\Theta_-)}, \tag{25}$$

$$\psi_3^f(s_3, t_3) = e^{-\alpha_3 s_3 - i\sqrt{E+\alpha^2_3}\, t_3} = e^{i\sqrt{E}\, r\cos(\varphi+\Phi+\Theta_-)}, \tag{26}$$

$$\psi_1^i(s_1, t_1) = e^{-\alpha_1 s_1 - i\sqrt{E+\alpha^2_1}\, t_1} = e^{-i\sqrt{E}\, r\cos(\varphi-\Phi+\Theta_+)}, \tag{27}$$

$$\psi_1^f(s_1, t_1) = e^{-\alpha_1 s_1 + i\sqrt{E+\alpha^2_1}\, t_1} = e^{i\sqrt{E}\, r\cos(\varphi-\Phi-\Theta_+)}. \tag{28}$$

The initial and final state functions of channel 0 are the plane wave functions:

$$\psi_0^i(r, \varphi) = e^{-ikr\cos(\varphi-\varphi')},$$
$$\psi_0^f(r, \varphi) = e^{ikr\cos(\varphi-\varphi')}, \tag{29}$$

where φ' is related in the following way to p_i and q_i defined by (3):

$$p_3 = -k\sin(\Phi+\varphi') \qquad p_1 = -k\sin(\Phi-\varphi'),$$
$$q_3 = -k\cos(\Phi+\varphi'), \qquad q_1 = k\cos(\Phi-\varphi'). \tag{30}$$

3. The solution of the scattering problem

Equation (7) with the boundary conditions (7.1) and (7.2) was solved by Malyuzhinetz [2], [3]. The equation should be supplemented by two further conditions guaranteeing uniqueness of the solution, namely:

1) $\Psi(r, \varphi)$ is bounded for $r = 0$;
2) The part of $\Psi(r, \varphi)$ resulting from the subtraction of the incident wave and of the waves reflected from the boundaries $\varphi = \pm\Phi$ is bounded at infinity. Malyuzhinetz represents the solution of Eq. (7) in the form of the generalized Fourier transform

$$\Psi(r, \varphi) = \frac{1}{2\pi i}\int_\gamma e^{-i\sqrt{E}\, r\cos z} s(z+\varphi)dz, \tag{31}$$

where γ is the Sommerfeld contour in the complex z plane. It is described in I for the both cases of $\sqrt{E} = k$ real, $k > 0$ and of $\sqrt{E}$ purely imaginary with Im $\sqrt{E} > 0$.

The method of calculation of the transform $s(z)$ of the Sommerfeld integral (31) is described in [11] and [12]. The function $s(z)$ is written in terms of the special meromorphic functions $M_\Phi(z)$, the definitions and the properties of which are given in the Appendix. The solution $\Psi(r, \varphi)$ written in terms of those special functions takes the following form:

$$\Psi(r, \varphi) = \frac{1}{2\pi i}\int_\gamma e^{-i\sqrt{E}\, r\cos z}\, \frac{\frac{\pi}{2\Phi}\cos\frac{\pi\varphi_0}{2\Phi}}{\sin\frac{\pi z}{2\Phi} - \sin\frac{\pi\varphi_0}{2\Phi}} \cdot F(z)\cdot[F(\varphi_0)]^{-1}dz, \tag{32}$$

where

$$F(z) = M_\Phi\left(z+\Phi+\frac{\pi}{2}-\Theta_+\right)M_\Phi\left(z-\Phi-\frac{\pi}{2}+\Theta_-\right)M_\Phi\left(z+\right.$$

$$\left.+\Phi-\frac{\pi}{2}+\Theta_+\right)M_\Phi\left(z-\Phi+\frac{\pi}{2}-\Theta_-\right), \tag{33}$$

and $\Psi(r, \varphi)$ is the solution of the scattering problem corresponding to the initial state $\exp\{-i\sqrt{E}r\cos(\varphi-\varphi_0)\}$. From (25) and (27) we see that when the initial states are in channel 3 or 1, φ_0 is complex and

$$\varphi_0 = -\Phi+\Theta_- \quad \text{for channel 3,} \tag{34}$$

$$\varphi_0 = \Phi-\Theta_+ \quad \text{for channel 1.} \tag{35}$$

Using the diffraction theory language we say that the surface wave is coming from infinity along the boundary $\varphi = -\Phi$ or along $\varphi = \Phi$.

In the case of scattering of three free particles, the initial state is given by the first expression (29), with $\varphi_0 = \varphi'$ being real and

$$-\Phi < \varphi_0 < \Phi. \tag{36}$$

As it was described in I the physically interesting quantities for the scattering problem under consideration may be deduced from studying the asymptotic behaviour of the solution (32) for large values of r. For positive values of E the asymptotic expression for the solution (32) takes on the form

$$\Psi(r, \varphi) \cong e^{-ikr\cos(\varphi-\varphi_0)} + \text{the waves reflected from the}$$

$$\text{boundaries } + C_- e^{ikr\cos(\varphi+\Phi+\Theta_-)} + C_+ e^{ikr\cos(\varphi-\Phi-\Theta_+)} +$$

$$+ f(\varphi, \varphi_0)(2\pi kr)^{-\frac{1}{2}} e^{i(kr+\pi/4)} \quad \text{for} \quad kr \gg 1. \tag{37}$$

The coefficients C_- and C_+ can be interpreted as the probability amplitudes for the formation of the bound states of the subsystems 3 and 1 respectively in the course of the scattering, and the coefficient $f(\varphi, \varphi_0)$ — as the probability amplitude for the free departure of three particles, as explained in Section 5 on the basis of the configuration space scattering theory. The explicit expressions for $C_\pm$ and $f(\varphi, \varphi_0)$ were written down in paper I. By substituting $\sqrt{E}$ in place of k in the expression (37) and taking into account the behaviour of Θ_+ and Θ_- as functions of E as it was described above in Section 3 we obtain the expression describing the asymptotic behaviour of $\Psi(r, \varphi)$ in all three energy regions I–IIII. The channel which is closed in a given energy region will be represented in this expression by an exponentially vanishing term.

Below in Sections 4 and 7 we consider separately the particular case of the BCM in which the interaction between the pair (2,3) is of the hard-core type, and the interaction between the pair (1,2) is still of the type described by (1). It is evident that this model can be treated as a particular case of the model described above, when we study first the behaviour of the two body solutions corresponding to the potential (1) for large values of $|\alpha|$. In the limit $1/|\alpha| = 0$ this boundary condition goes over into the condition

$$\Psi(0) = 0, \tag{38}$$

representing the one-dimensional hard core interaction. Hence, making the transition $|\alpha_1| \to \infty$ in our three body model described previously we obtain the model in which the condition (4.1) remains unchanged and (4.2) is substituted by

$$\Psi(s, t) = 0 \quad \text{for} \quad s_1 = 0. \tag{39}$$

From the second expression of (11) we see that the transition $|\alpha_1| \to \infty$ corresponds to:

$$\mathrm{Im}\, \Theta_+ \to -\infty. \tag{40}$$

In Sections 4 and 7 we apply the limiting procedure (40) to the general expressions for probability amplitudes and cross-sections obtaining in result the corresponding quantities for the hard core model.

4. The probability amplitudes

Here we collect the explicit expressions for the amplitudes $C_\pm$ and $f(\varphi, \varphi_0)$ with specified initial situations. The results are obtained by substituting φ_0 of the form (34), (35) or (36) into the expressions (36), (37) of I, and then by using trigonometric transformations and functional relations for the functions $M_\Phi(z)$.

All the amplitudes may be collected in the following matrix form:

$$\begin{pmatrix} C_-^3 & C_+^3 & f^3(\varphi) \\ C_-^1 & C_+^1 & f^1(\varphi) \\ C_-^0(\varphi_0) & C_+^0(\varphi_0) & f^0(\varphi, \varphi_0) \end{pmatrix}, \tag{41}$$

where the superscript denotes the initial channel and the last row and column are understood to be the sets of quantities with continuously changing variable φ_0 and φ respectively, with $|\varphi|, |\varphi_0| < \Phi$.

For the general case of the model, with finite values of α_1 and α_3 the expressions for the amplitudes are:

$$C_-^3 = \mathrm{ctg}\, \Theta_- \frac{\cos m/2(\Theta_+ - \Theta_- - \pi)}{\cos^2 m/2(\Theta_+ - \Theta_-) \cos m/2(\Theta_+ + \Theta_-) \sin m/2(2\Theta_- + \pi)} \times$$

$$\times [M_\Phi(\pi/2)]^{-6} [M_\Phi(\pi/2 + \Theta_+ - \Theta_-) M_\Phi(\pi/2 + \Theta_+ + \Theta_-) M_\Phi(2\Phi + \pi/2 + 2\Theta_-)]^2, \tag{42}$$

where

$$m = \pi/2\Phi; \tag{43}$$

$$C_+^3 = \frac{\sin m\pi/2\, \mathrm{ctg}\, \Theta_-}{\cos m/2(\Theta_+ - \Theta_-) \cos m/2(\Theta_+ + \Theta_-) \cos m/2(\Theta_+ + \Theta_- + \pi)} \times$$

$$\times [M_\Phi(\pi/2)]^{-4} [M_\Phi(\pi/2 + \Theta_+ + \Theta_-)]^2 M_\Phi(2\Phi + \pi/2 + 2\Theta_+) M_\Phi(2\Phi + \pi/2 + 2\Theta_-), \tag{44}$$

$$f^3(\varphi) = \frac{-4\sqrt{m}\,\mathrm{ctg}\,\Theta_-\,\sin m\pi/2\,\cos m\varphi\,\cos m/2(\Theta_+-\Theta_--\pi)}{\cos m/2(\Theta_+-\Theta_-)\,[\cos m(\Theta_-+\pi)+\sin m\varphi]\,[\cos m\Theta_-+ +\sin m\varphi]\,[\cos m\Theta_+-\sin m\varphi]}\times$$

$$\times[M_\Phi(\pi/2)]^{-7}M_\Phi(\pi/2+\Theta_++\Theta_-)M_\Phi(2\Phi+\pi/2+2\Theta_-)M_\Phi(\pi/2+\Theta_+-\Theta_-)\cdot L(\varphi), \quad (45)$$

where

$$L(\varphi) = M_\Phi(\varphi+\Phi-\pi/2-\Theta_+)M_\Phi(\varphi+\Phi+\pi/2+\Theta_+)M_\Phi(\varphi-\Phi-\pi/2-\Theta_-)M_\Phi(\varphi- -\Phi+\pi/2+\Theta_-,) \quad (46)$$

$$f^0(\varphi, \varphi_0) = \frac{4m\sin m\pi/2\,\cos m\varphi\,\cos m\varphi_0[M_\Phi(\pi/2)]^{-8}\cdot L(\varphi)\cdot L(\varphi_0)}{[\cos m(\varphi-\varphi_0)-\cos m\pi]\,[\cos m(\varphi+\varphi_0)+\cos m\pi]\,(\cos m\Theta_+-\sin m\varphi_0)}\times$$

$$\times\frac{1}{(\cos m\Theta_-+\sin m\varphi_0)\,(\cos m\Theta_+-\sin m\varphi)\,(\cos m\Theta_-+\sin m\varphi)}\times$$

$$\times\{(\sin m\varphi+\sin m\varphi_0)\,[\cos m(\Theta_+-\pi/2)-\cos m(\Theta_--\pi/2)]+$$

$$+2\cos m\pi/2[\cos m(\Theta_+-\pi/2)\cos m(\Theta_--\pi/2)-\sin m\varphi\,\sin m\varphi_0-\sin^2 m\pi/2]\} \quad (47)$$

$$C^1_+ = \frac{\mathrm{ctg}\,\Theta_+\,\cos m/2(\Theta_+-\Theta_-+\pi)}{\cos^2 m/2(\Theta_+-\Theta_-)\cos m/2(\Theta_++\Theta_-)\sin m/2(2\Theta_++\pi)}\times$$

$$\times[M_\Phi(\pi/2)]^{-6}[M_\Phi(\pi/2+\Theta_--\Theta_+)M_\Phi(\pi/2+\Theta_++\Theta_-)M_\Phi(2\Phi+\pi/2+2\Theta_+)]^2, \quad (48)$$

$$f^1(\varphi) = \frac{-4\sqrt{m}\,\mathrm{ctg}\,\Theta_+\,\sin m\pi/2\,\cos m\varphi\,\cos m/2(\Theta_+-\Theta_-+\pi)}{\cos m/2(\Theta_+-\Theta_-)\,[\cos m(\pi+\Theta_+)-\sin m\varphi]\,[\cos m\Theta_+- -\sin m\varphi]\,[\cos m\Theta_-+\sin m\varphi]}\times$$

$$\times[M_\Phi(\pi/2)]^{-7}M_\Phi(\pi/2+\Theta_++\Theta_-)M_\Phi(2\Phi+\pi/2+2\Theta_+)M_\Phi(\pi/2+ +\Theta_--\Theta_+)\cdot L(\varphi). \quad (49)$$

Direct calculations give

$$C^1_- = \mathrm{tg}\,\Theta_-\cdot\mathrm{ctg}\,\Theta_+\cdot C^3_+, \quad (50)$$

$$C^0_-(\varphi_0) = -2\,\mathrm{tg}\,\Theta_-\cdot f^3(\varphi=\varphi_0), \quad (51)$$

$$C^0_+(\varphi_0) = -2\,\mathrm{tg}\,\Theta_+\cdot f^1(\varphi=\varphi_0). \quad (52)$$

These expressions were obtained from the expressions given in I by extensive rearrangement in which physical poles and zeros were made to appear in the trigonometric parts of the expressions[1].

[1] Expressions (47) and (56) lose sense for those values of real φ which correspond to the poles of the above expressions. For these singular directions φ, the asymptotic representation (37) must be replaced by a more general one (for the wedge problem with the boundary condition $\Psi(r, \pm\Phi) = 0$ see [13]).

As it was described in the last part of Section 3 we may apply the limiting procedure (40) to the above expressions obtaining the quantities characterizing the model in which the interaction between the pair (2, 3) is described by the condition (39). The table of amplitudes for this case can be obtained from (41) by cancelling off the middle row and column. Barred symbols correspond to this particular case. Making use of the expression (A.11) for the asymptotic behaviour of $M_{\Phi}(z)$ for $|\mathrm{Im}\, z| \to \infty$ and of the relation (A. 9) we obtain the following results:

$$\bar{C}^3_- = \frac{\mathrm{ctg}\,\Theta}{\sin m(\Theta+\pi/2)}\,[M_{\Phi}(\pi/2)]^{-2}\cdot[M_{\Phi}(2\Phi+\pi/2+2\Theta)]^2, \tag{53}$$

$$\bar{f}^3(\varphi) = \frac{-2\sqrt{m}\,\mathrm{ctg}\,\Theta\cdot\sin m\pi/2\cdot\cos m\varphi}{(\cos m\Theta+\sin m\varphi)\,[\cos m(\Theta+\pi)+\sin m\varphi]}\,[M_{\Phi}(\pi/2)]^{-3}\times$$

$$\times M_{\Phi}(2\Phi+\pi/2+2\Theta)\cdot\bar{L}(\varphi), \tag{54}$$

$$\bar{f}^0(\varphi,\varphi_0) = \frac{4m\cdot\sin m\pi/2\cdot\cos m\varphi\cdot\cos m\varphi_0}{[\cos m(\varphi-\varphi_0)-\cos m\pi]\,[\cos m(\varphi+\varphi_0)+\cos m\pi]}\times$$

$$\times\frac{\sin m\varphi+\sin m\varphi_0+2\cos m\pi/2\cos m(\Theta-\pi/2)}{(\sin m\varphi_0+\cos m\Theta)\,(\sin m\varphi+\cos m\Theta)}\,[M_{\Phi}(\pi/2)]^{-4}\cdot\bar{L}(\varphi)\cdot\bar{L}(\varphi_0), \tag{55}$$

$$\bar{C}^0_-(\varphi_0) = -2\,\mathrm{tg}\,\Theta\cdot\bar{f}^3(\varphi=\varphi_0), \tag{56}$$

where

$$\Theta\equiv\Theta_-,\qquad \bar{L}(\varphi) = M_{\Phi}(\varphi-\Phi-\pi/2-\Theta)M_{\Phi}(\varphi-\Phi+\pi/2+\Theta).$$

From the above expressions we see immediately that

$$f^1(\varphi)=f^3(\varphi)=f^0(\varphi,\varphi_0)=\bar{f}^3(\varphi)=\bar{f}^0(\varphi,\varphi_0)=0\quad\text{for }\varphi=\pm\Phi,$$

$$C^0_{\mp}(\varphi_0)=f^0(\varphi,\varphi_0)=\bar{C}^0_{\mp}(\varphi_0)=\bar{f}^0(\varphi,\varphi_0)=0\quad\text{for}\quad\varphi_0=\pm\Phi. \tag{57}$$

For such values of Φ that $m=2n$, where n is an integer, no dissociation nor rearrangement takes place as can be seen from Eq. (44), (45), (47) which contain the factor $\sin m\pi/2$. For such a case the full solution of the scattering problem is given by the sum of the incident wave and the reflected waves which are of the same nature as the incident wave.

For $m=2n+1$ no such degeneration of the solution independent of the $\Theta_{\pm}$ takes place. If we, however, put further in the expressions for the general case $\Theta_+=\Theta_-$ (*i.e.* $\alpha_1=\alpha_3$) we find that the only nonzero amplitude is that of the rearrangement process. When the initial channel is 1 (or 3) the reflected wave leaving the reaction center is in channel 3 (or 1). When the initial state is a plane wave $\exp\{-ikr\cos(\varphi-\varphi_0)\}$ there is no truly cylindrical wave and the plane wave leaving the reaction center is $\exp\{ikr\cos(\varphi+\varphi_0)\}$. The simplest case corresponding to the group of solutions described above is the system of three particles with equal masses and equal constants $\alpha_1=\alpha_3$ for which the Faddeev equations in the momentum representation were studied in paper [5].

From the properties (A.6) of the function $M_\Phi(z)$ we see that when $m = 2k/l$, where k and l are natural numbers and the whole fraction is rereducible, the expressions for the amplitudes may be written in terms of elementary functions. The model discussed by Nussenzweig in [6] belongs to this group of solutions with $m = 4/3$ and one of the two-body interactions of the hard core type.

In order to study further the analytic behaviour of the amplitudes as functions of α_1, α_3, E and mass ratios, one has to substitute the expressions relating $\Theta_\pm$ to these variables into the expressions for the amplitudes and to employ the definition of the special function $M_\Phi(z)$ given in the Appendix. We shall however confine ourselves to the discussion of the reversibility rules, given in the next section.

5. *Configuration space treatment of the three particle problem*

We base our considerations on the Gerjuoy's method of approach to n-body collision theory [14]–[16]. The fundamental theorem in this procedure is the multidimensional Green theorem, which — when applied to the solutions of the Schrödinger equation — states that given two solutions Ψ_1 and Ψ_2 of the same Schrödinger equation:

$$\left[\sum_{i=1}^{n}\left(-\frac{\hbar^2}{m_i}\nabla_{r_i}^2\right)+V(\vec{r})-E\right]\Psi(\vec{r}) = 0 \tag{58}$$

for n particles with masses m_i and coordinates $\vec{r}_i$,

$$\vec{r} = [\vec{r}_1, \ldots, \vec{r}_n]$$

then

$$\int dSJ(\Psi_2, \Psi_1) = \frac{1}{i\hbar}\int d\vec{S}\cdot\vec{W}[\Psi_2, \Psi_1] = 0, \tag{59}$$

where the integral is over the surface at infinity in the $\vec{r}$ space; $d\vec{S} = \vec{n} \neq dS$, where $\vec{n}$ is the unit outward normal to the surface; $\vec{W}$ is the vector in the $\vec{r}$ space,

$$\vec{W} = [\vec{W}_1, \ldots, \vec{W}_n],$$

with the components:

$$\vec{W}_i[\Psi_2, \Psi_1] = \frac{\hbar^2}{2m_i}(\Psi_2\nabla_{r_i}\Psi_1 - \Psi_1\nabla_{r_i}\Psi_2). \tag{60}$$

The vector operator $1/i\hbar\vec{W}$ is the quantum-mechanical current operator for the n-particle system in the $\vec{r}$ space. Gerjuoy has shown [14], [15] that the solution Ψ of the n-particle scattering problem can be uniquely determined by the boundary condition for large values of r stating that the scattered wave $\psi_{sc} = \Psi - \psi_{inc}$ has the same asymptotic behaviour as the Green function G^+ of the total Hamiltonian, in the following sense

$$\int_\infty dS'J[G^+(\vec{r}, \vec{r}'), \Psi_{sc}(\vec{r}')] = 0 \quad \text{for all } \vec{r}. \tag{61}$$

In equation (59) each direction $\vec{n}$ at infinity corresponds to a possible formation of a definite class of aggregates of the n particle system. Substituting $\Psi_1 = \Psi$ and $\Psi_2 = \Psi^*$ in (59) and calculating the surface integral at infinity as a sum of contributions from all elements $d\vec{S}$ corresponding to the formation af all energetically possible aggregates of the system described by the wave function Ψ, we obtain the probability current conservation rules expressed in terms of the cross-sections of the scattering processes. When Ψ_1 and Ψ_2 in (59) are two scattering solutions with different initial channels we obtain with the aid of (59) the reciprocity rules relating the probability amplitudes of the mutually reciprocal processes.

For the Schrödinger equation (4) written in terms of the coordinates r, φ as given by (7), equation (59) takes the following form:

$$\lim_{r\to\infty} \int_{-\Phi}^{\Phi} d\varphi r J(\Psi_2, \Psi_1) = 0, \tag{62}$$

where

$$J(\Psi_2, \Psi_1) = \frac{1}{i}\left[\Psi_2 \frac{\partial}{\partial r}\Psi_1 - \Psi_1 \frac{\partial}{\partial r}\Psi_2\right] \tag{63}$$

and Ψ_1, Ψ_2 are two solutions corresponding to the same value of energy E, different from zero in the region $|\varphi| < \Phi$;

$$\Psi_1, \Psi_2 = \Psi^i \quad i = 0, 1, 3, \tag{64}$$

where the superscript i denotes the initial channel of the solution Ψ_i.

In order to calculate the surface integral (62) we employ the asymptotic expression (37) which shall correspond to a determined initial channel 1,3 or 0 when we put

$$\varphi_0 = \Phi - \Theta_+, \ \varphi_0 = -\Phi + \Theta_-, \text{ or } \varphi_0 \text{ real, } |\varphi_0| < \Phi.$$

Let us put in (62):

$$\Psi_1 = \Psi^3; \ \Psi_2 = \Psi^1. \tag{65}$$

The only nonzero contributions to the surface integral (62) at infinity are those from the elements $rd\varphi$ in the directions $\pm\Phi$, or — in other words — those in the directions $s_1 = 0$, $s_3 = 0$. Expressing the state functions of channels 1 and 3 in terms of the coordinates (s, t), as given by the expressions (25)–(28) we obtain in the limit of large r:

$$-2C_-^1 \cdot \mathrm{Im}\left[e^{-i\sqrt{E+\alpha^2_3}\,t_3} \frac{d}{dt_3} e^{i\sqrt{E+\alpha^2_3}\,t_3}\right] \cdot \int_{-\infty}^{0} e^{-2\alpha_3 s_3} ds_3 +$$

$$+2C_+^3 \cdot \mathrm{Im}\left[e^{-i\sqrt{E+\alpha^2_1}\,t_1} \frac{d}{dt_1} e^{i\sqrt{E+\alpha^2_1}\,t_1}\right] \cdot \int_{-\infty}^{0} e^{-2\alpha_1 s_1} ds_1 = 0. \tag{66}$$

In result, employing also the relations (10) and (11), we get

$$\operatorname{ctg}\Theta_{-}\cdot C_{-}^{1}=\operatorname{ctg}\Theta_{+}\cdot C_{+}^{3}, \tag{67}$$

in agreement with (50).

In order to obtain the equation relating the amplitudes for the recombination and dissociation processes we put

$$\Psi_1=\Psi^3,\qquad \Psi_2=\Psi^0,\qquad 0<E<\infty, \tag{68}$$

in the integral (62).

We use the following asymptotic expression for the incident plane wave [19]

$$e^{-ikr\cos(\varphi-\varphi_0)}\xrightarrow[r\to\infty||\varphi]{}\sqrt{\frac{2\pi}{kr}}\,e^{-i\pi/4}\{e^{ikr}\delta(\varphi-\varphi_0+\pi)+ie^{-ikr}\delta(\varphi-\varphi_0)\}. \tag{69}$$

We note that the support of the first δ-function in the expression (69) lies outside the region $|\varphi|<\Phi$.

For Ψ_1 and Ψ_2 given by (68) the only nonzero contributions at infinity to the integral (62) are:

$$J^0 r d\varphi=J\left[\sqrt{\frac{2\pi}{kr}}\,ie^{-i(kr+\pi/4)}\delta(\varphi-\varphi_0),f^3(\varphi)\frac{1}{\sqrt{2\pi kr}}\,e^{i(kr+\pi/4)}\right]rd\varphi \tag{70}$$

along the direction $\varphi=\varphi_0$ and — in terms of the coordinate (s_3,t_3):

$$J^3 r d\varphi=J[C_{-}^{0}(\varphi_0)e^{-\alpha_3 s_3-i\sqrt{E+\alpha^2{}_3}\,t_3},\ e^{-\alpha_3 s_3+i\sqrt{E+\alpha^2{}_3}\,t_3}]ds_3 \tag{71}$$

along the direction $s_3=0$.

The "surface wave" terms $\exp\{-\alpha_3 s_3\pm i\sqrt{E+\alpha_3^2}\,t_3\}$ and $\exp\{-\alpha_1 s_1\pm i\sqrt{E+\alpha_1^2}\,t_1\}$ and the dissociation terms do not overlap because the dissociation terms vanish on the boundaries $\varphi=\pm\Phi$.

When calculating the contribution from (71) to the integral (62) we employ the fact that $\exp\{-\alpha_3 s_3\}$ is exponentially decreasing for $s_3\to-\infty$ thus allowing us to extend the s_3 integration limits to $(-\infty,0)$.

Hence we obtain:

$$\int_{-\Phi}^{\Phi}J^0 r d\varphi+\int_{-\infty}^{0}J^3 ds_3=2if^3(\varphi=\varphi_0)+\frac{\sqrt{E+\alpha_3^2}}{\alpha_3}\,C_{-}^{0}(\varphi_0)=0, \tag{72}$$

$$C_{-}^{0}(\varphi_0)=-2\operatorname{tg}\Theta_{-}\cdot f^3(\varphi=\varphi_0). \tag{73}$$

In a similar way we get:

$$C_{+}^{0}(\varphi_0)=-2\operatorname{tg}\Theta_{+}\cdot f^1(\varphi=\varphi_0) \tag{74}$$

in agreement with (51) and (52).

As a consequence of employing (62) for two solutions corresponding to the incidence of the plane waves in the directions φ_1 and φ_2 respectively we obtain the rule of symmetry for the amplitude $f^0(\varphi, \varphi_0)$:

$$f^0(\varphi_1, \varphi_2) = f^0(\varphi_2, \varphi_1), \tag{75}$$

which is easily seen to be satisfied by the expressions (47) and (55).

The flux conservation rules can be derived from the general relation

$$\lim_{r\to\infty} \int_{-\Phi}^{\Phi} d\varphi r J(\Psi^*, \Psi) = 0. \tag{76}$$

We define the cross-section for a given scattering process as the probability for such a process to take place divided by the incident probability flux. According to Gerjuoy [14], [15] the corresponding probability currents may be calculated as contributions to the integral (76) from the directions φ in infinity corresponding to the formation of the appriopriate aggregate in the three body system. The directions $\varphi = \pm\Phi$ correspond to the formation of the pairs (1,2) and (2,3) and the direction $\varphi = \varphi'$, $|\varphi'| < \Phi$ corresponds to the free departure of three particles, with φ' playing the role of their polar momentum coordinate as given by (30).

Employing (76) for $\Psi = \Psi^3$ and making use of the definitions of cross-sections given above we obtain

$$\sigma^3_{\mathrm{el}}(E) + \sigma^3_{\mathrm{rearr}}(E) + \sigma^3_{\mathrm{diss}}(E) = 1, \tag{77}$$

where

$$\sigma^3_{\mathrm{el}}(E) = |C^3_-|^2, \tag{78}$$

$$\sigma^3_{\mathrm{rearr}}(E) = \frac{\alpha_3}{\sqrt{E+\alpha_3^2}} \cdot \frac{\sqrt{E+\alpha_1^2}}{\alpha_1} |C^3_+|^2 = \mathrm{tg}\,\Theta_- \cdot \mathrm{ctg}\,\Theta_+ |C^3_+|^2, \tag{79}$$

$$\sigma^3_{\mathrm{diss}}(E) = \int_{-\Phi}^{\Phi} \sigma^3_{\mathrm{diss}}(E, \varphi) d\varphi, \tag{80}$$

$$\sigma^3_{\mathrm{diss}}(E, \varphi) = \frac{-\alpha_3}{\sqrt{E+\alpha_3^2}} \cdot \frac{1}{\pi} |f^3(\varphi)|^2 = \frac{1}{\pi} i\,\mathrm{tg}\,\Theta_- |f^3(\varphi)|^2. \tag{81}$$

For the values of E below a given threshold, the corresponding cross-section does not contribute to the sum in (77). In a similar way we can define the cross-sections for the system described by the function Ψ^1:

$$\sigma^1_{\mathrm{el}}(E) = |C^1_+|^2, \tag{82}$$

$$\sigma^1_{\mathrm{rearr}}(E) = \mathrm{tg}\,\Theta_+ \cdot \mathrm{ctg}\,\Theta_- \cdot |C^1_-|^2, \tag{83}$$

$$\sigma^1_{\mathrm{diss}}(E, \varphi) = \frac{i}{\pi} \mathrm{tg}\,\Theta_+ \cdot |f^1(\varphi)|^2, \tag{84}$$

$$\sigma^1_{\mathrm{el}}(E) + \sigma^1_{\mathrm{rearr}}(E) + \int_{-\Phi}^{\Phi} \sigma^1_{\mathrm{diss}}(E, \varphi) d\varphi = 1. \tag{85}$$

From (67), (79) and (83) we get

$$\sigma^1_{\text{rearr}}(E) = \sigma^3_{\text{rearr}}(E). \tag{86}$$

In the case of an initial state of three free particles we find the usual difficulties when trying to define the cross-sections. There are plane waves incoming and outgoing to infinity [16]. We give here only the probabilities of occurrence of recombination and of "elastic" scattering resulting from the encounter of three free particles described by the plane wave $\exp\{-ikr\cos(\varphi-\varphi_0)\}$.

The probabilities for the recombination processes are

$$P^0_3(k^2, \varphi_0) = \frac{\sqrt{k^2+\alpha_3^2}}{-\alpha_3}|C^0_-(\varphi_0)|^2 = -i\operatorname{ctg}\Theta_-|C^0_-(\varphi_0)|^2, \tag{87}$$

$$P^0_1(k^2, \varphi_0) = \frac{\sqrt{k^2+\alpha_1^2}}{-\alpha_1}|C^0_+(\varphi_0)|^2 = -i\operatorname{ctg}\Theta_+|C^0_+(\varphi_0)|^2. \tag{88}$$

The probability for elastic scattering of three particles into the angle element $d\varphi$ is

$$P^0(k^2; \varphi, \varphi_0) = \frac{1}{\pi}|f^0(\varphi, \varphi_0)|^2 d\varphi. \tag{89}$$

It should be mentioned that we have also obtained the relations of the type (73)–(75) by expressing Ψ as the solution of the appropriate Lippmann-Schwinger equation and by imposing next the time-reversal invariance for the system under consideration. However, because of the highly singular character of the potentials we use, the route *via* the Green theorem seems to be a more natural one.

In the next section we derive the expressions for the cross-sections and use them to illustrate the flux conservation rules.

6. Cross-sections. The general case

In this section we show that in distinction to the amplitudes the cross-sections for all possible scattering processes in our model may be expressed in terms of elementary functions in a relatively simple form.

We limit ourselves to the case when initially one pair is bounded. For the scattering problem in which initially three particles move freely it is not possible to define the cross--sections in the same manner, as we do it for the bound state scattering [14], [16]. However, also in this last case the probabilities of all possible scattering events are also expressed in terms of elementary functions, as is shown below.

Let us consider first the energy region *I*, for which the total energy E is below the rearrangement process threshold. In this region Θ_+ is real and Θ_- is given by (15), (16). Using this fact, employing (A.13)–(A.15), (42) and the definition (78) of the elastic cross--section we obtain that

$$\sigma^3_{\text{el}}(-n^2) = 1 \tag{90}$$

in agreement with the flux conservation rule.

In a similar way using (42), (44), (48), (50) and the functional relations of Appendix we find that the elastic and exchange cross-sections in the energy region II:

$$-\alpha_1^2 < E = -l^2 < 0$$

are expressed as follows:

$$\sigma^3_{\rm el}(-l^2) = \frac{\text{ch}^2\, m/2(\Theta^l_+ + \Theta^l_-)\cdot[\text{ch}^2\, m/2(\Theta^l_+ - \Theta^l_-) - \sin^2 m\pi(2]}{\text{ch}^2\, m/2(\Theta^l_+ - \Theta^l_-)\cdot[\text{ch}^2\, m/2(\Theta^l_+ + \Theta^l_-) - \sin{}^2 m\pi/2]} =$$

$$= \frac{\text{ch}^2\, m/2(\Theta^l_+ - \Theta^l_-) - \sin^2 m\pi/2}{\text{ch}^2\, m/2(\Theta^l_+ - \Theta^l_-)}\left[1 + \frac{\sin^2 m\pi/2}{\text{ch}^2\, m/2(\Theta^l_+ + \Theta^l_-) - \sin^2 m\pi/2}\right] =$$

$$= \sigma^1_{\rm el}(-l^2), \tag{91}$$

$$\sigma^3_{\rm rearr}(-l^2) = \frac{\sin^2 m\pi/2\cdot \text{sh}\, m\Theta^l_+ \cdot \text{sh}\, m\Theta^l_-}{\text{ch}^2\, m/2(\Theta^l_+ - \Theta^l_-)\cdot[\text{ch}^2\, m/2(\Theta^l_+ + \Theta^l_-) - \sin^2 m\pi/2]} =$$

$$= \sigma^1_{\rm rearr}(-l^2), \tag{92}$$

where $\Theta^l_\pm$ are given by (19) and (20).

Adding (91) and (92) we get

$$\sigma_{\rm el}(-l^2) + \sigma_{\rm rearr}(-l^2) = 1. \tag{93}$$

For the energy region over the break-up threshold we obtain

$$\sigma^3_{\rm el}(k^2) = \frac{\text{sh}^2\, m\Theta^k_-[\text{ch}^2\, m/2(\Theta^k_+ - \Theta^k_-) - \sin^2 m\pi/2]}{\text{ch}^2\, m/2(\Theta^k_+ - \Theta^k_-)\cdot[\text{sh}^2\, m\Theta^k_- + \sin^2 m\pi/2]} =$$

$$= \frac{\text{ch}^2\, m/2(\Theta^k_+ - \Theta^k_-) - \sin^2 m\pi/2}{\text{ch}^2\, m/2(\Theta^k_+ - \Theta^k_-)}\left[1 - \frac{\sin^2 m\pi/2}{\text{sh}^2\, m\Theta^k_- + \sin^2 m\pi/2}\right], \tag{94}$$

$$\sigma^1_{\rm el}(k^2) = \frac{\text{ch}^2\, m/2(\Theta^k_+ - \Theta^k_-) - \sin^2 m\pi/2}{\text{ch}^2\, m/2(\Theta^k_+ - \Theta^k_-)}\left[1 - \frac{\sin^2 m\pi/2}{\text{sh}^2\, m\Theta^k_+ + \sin^2 m\pi/2}\right], \tag{95}$$

$$\sigma^3_{\rm rearr}(k^2) = \frac{\sin^2 m\pi/2\cdot \text{sh}\, m\Theta^k_- \cdot \text{sh}\, m\Theta^k_+}{\text{ch}^2\, m/2(\Theta^k_+ - \Theta^k_-)\,[\text{ch}^2\, m/2(\Theta^k_+ + \Theta^k_-) - \sin^2 m\pi/2]} = \sigma^1_{\rm rearr}(k^2), \tag{96}$$

$$\sigma^3_{\rm diss}(k^2, \varphi) = \frac{4m}{\pi}\sin^2 m\pi/2\ \text{sh}\, m\Theta^k_- \frac{\text{ch}\, m/2(\Theta^k_+ + \Theta^k_-)}{\text{ch}\, m/2(\Theta^k_+ - \Theta^k_-)}\times$$

$$\times[\text{ch}^2\, m/2(\Theta^k_+ - \Theta^k_-) - \sin^2 m\pi/2]\times$$

$$\times\frac{\cos^2 m\varphi}{[(\cos m\pi\ \text{ch}\, m\Theta^k_- + \sin m\varphi)^2 + \sin^2 m\pi\ \text{sh}^2\, m\Theta^k_-]\,(\text{ch}\, m\Theta^k_+ - \sin m\varphi)(\text{ch}\, m\Theta^k_- + \sin m\varphi)}, \tag{97}$$

where $\Theta^k_\pm$ are given by (23) and (24) and the differential break-up cross-section $\sigma^3_{\text{dis}}(k^2, \varphi)$ is defined by Eq. (81). The expression for $\sigma^1_{\text{diss}}(k^2, \varphi)$ is obtained from (97) by simple exchange of Θ^k_+ and Θ^k_- and by substituting $-\varphi$ for φ. According to (80) the total break-up cross-section $\sigma_{\text{diss}}(k^2)$ is to be calculated by integrating $\sigma_{\text{diss}}(k^2, \varphi)$ with respect to φ over the interval $(-\Phi, \Phi)$. The very lengthy integration of (97) could be performed by the method of residues; however we find $\sigma_{\text{diss}}(k^2)$ with the aid of the conservation laws. We found that this results agree with the explicit calculation for the particular case of $\Theta^k_+ \to \infty$ (see Section 7).

The total break-up cross-section calculated with the aid of the unitary condition (77) equals to

$$\sigma^3_{\text{diss}}(k^2) = \sin^2 m\pi/2 \, \frac{\text{ch}^2\, m/2(\Theta^k_+ - \Theta^k_-) - \sin^2 m\pi/2}{\text{ch}^2\, m/2(\Theta^k_+ - \Theta^k_-)} \times$$

$$\times \left[\frac{1}{\text{ch}^2\, m/2(\Theta^k_+ + \Theta^k_-) - \sin^2 m\pi/2} + \frac{1}{\text{sh}^2\, m\Theta^k_- + \sin^2 m\pi/2} \right]. \tag{98}$$

An expression for $\sigma^1_{\text{diss}}(k^2)$ is obtained from (98) by interchanging the indices $+$ and $-$. Employing (19), (20) and (23), (24) we may express the hyperbolic functions occurring in the expressions (91)–(98) in terms of α_1, α_3 and E in the following way:

$$\begin{matrix}\text{sh}\, m\Theta^l_- \\ \text{ch}\, m\Theta^l_-\end{matrix} = \frac{1}{2} \frac{(\sqrt{E+\alpha_3^2} - \alpha_3)^m \mp (-\sqrt{E+\alpha_3^2} - \alpha_3)^m}{(\sqrt{-E})^m}, \tag{99}$$

$$\begin{matrix}\text{sh}\, m\Theta^l_+ \\ \text{ch}\, m\Theta^l_+\end{matrix} = \frac{1}{2} \frac{(\sqrt{E+\alpha_1^2} - \alpha_1)^m \mp (-\sqrt{E+\alpha_1^2} - \alpha_1)^m}{(\sqrt{-E})^m}, \tag{100}$$

$$\text{ch}\, m/2(\Theta^l_+ \pm \Theta^l_-) = \tfrac{1}{2} \times$$

$$\times \frac{(\sqrt{E+\alpha_3^2} - \alpha_3)^{m/2} (\pm\sqrt{E+\alpha_1^2} - \alpha_1)^{m/2} + (-\sqrt{E+\alpha_3^2} - \alpha_3)^{m/2} (\mp\sqrt{E+\alpha_1^2} - \alpha_1)^{m/2}}{(-\sqrt{E})^m} \tag{101}$$

$$\text{for } -\alpha_1^2 < E = -l^2 < 0,$$

and

$$\begin{matrix}\text{sh}\, m\Theta^k_- \\ \text{ch}\, m\Theta^k_-\end{matrix} = \frac{1}{2} \frac{(\sqrt{E+\alpha_3^2} - \alpha_3)^m \mp (\sqrt{E+\alpha_3^2} + \alpha_3)^m}{(\sqrt{E})^m}, \tag{102}$$

$$\begin{matrix}\text{sh}\, m\Theta^k_+ \\ \text{ch}\, m\Theta^k_+\end{matrix} = \frac{1}{2} \frac{(\sqrt{E+\alpha_1^2} + \alpha_1)^m \mp (\sqrt{E+\alpha_1^2} + \alpha_1)^m}{(\sqrt{E})^m}, \tag{103}$$

$$\text{ch}\, m/2(\Theta^k_+ \pm \Theta^k_-) = \tfrac{1}{2} \times$$

$$\times \frac{(\sqrt{E+\alpha_3^2} - \alpha_3)^{m/2} (\sqrt{E+\alpha_1^2} \mp \alpha_1)^{m/2} + (\sqrt{E+\alpha_3^2} + \alpha_3)^{m/2} (\sqrt{E+\alpha_1^2} \pm \alpha_1)^{m/2}}{(\sqrt{E})^m} \tag{104}$$

$$\text{for } E = k^2 > 0.$$

$m = \pi/2\Phi$ is in general an irrational number. In all the above expressions, the bases of the exponential function z^m and $z^{m/2}$ are positive numbers with $\arg z = 0$, *i.e.* $z^m = e^{mLnz}$. The expressions for the recombination probabilities defined by (87) and (88) can be easily derived from the expression for dissociation cross-sections by employing the reversibility rules (73) and (74). The probability $P^0(\varphi, \varphi_0)$ for the elastic scattering of free particles as defined by (89) may be calculated with the use of expression (47).

Employing the result

$$\left| \frac{L(\varphi)}{[M_\Phi(\pi/2)]^4} \right|^2 = \frac{1}{4} (\text{ch } m\Theta^k_+ - \sin m\varphi)(\text{ch } m\Theta^k_- + \sin m\varphi)$$

we deduce that $P^0(\varphi, \varphi_0)$ can also be expressed in terms of elementary functions only. We note that $P^0(\varphi, \varphi_0)$ calculated in the manner described above is not integrable over $\varphi \in [-\Phi, \Phi]$.

From the results of this section we see that for m being an odd integer, the expressions for the cross-sections simplify considerably and, on the other hand, the scattering problem is still nontrivial.

7. *Cross-sections. The model with the hard core potential*

According to the discussion o Section 3, when we apply the limiting procedure

$$\text{Im } \Theta_+ \to \infty$$

or, in other words, when

$$\Theta^l_+ \to \infty \text{ for } E < 0 \text{ and } \Theta^k_+ \to \infty \text{ for } E > 0 \tag{105}$$

keeping Θ_- as function of E for constant α_3 we obtain the quantities corresponding to the model for which the condition (4.1) remains unchanged and condition (4.2) is substituted by (39).

In this section we apply this limiting process to the expressions for cross-sections σ^3 obtained in Section 6. The resulting quantities are denoted $\bar{\sigma}_{el}$ and $\bar{\sigma}_{diss}$.

Employing (91)–(97) we obtain

$$\bar{\sigma}_{\text{el}}(-l^2) = 1, \tag{106}$$

$$\bar{\sigma}_{\text{el}}(k^2) = \frac{\text{sh}^2\, m\Theta^k}{\text{sh}^2\, m\Theta^k + \sin^2 m\pi/2} = 1 - \frac{\sin^2 m\pi/2}{\text{sh}^2\, m\Theta^k + \sin^2 m\pi/2}, \tag{107}$$

$$\bar{\sigma}_{\text{diss}}(k^2, \varphi) = \frac{2m}{\pi} \sin^2 m\pi/2 \text{ sh } m\Theta^k \times$$

$$\times \frac{\cos^2 m\varphi}{(\cos m\pi \text{ ch } m\Theta^k + \sin m\varphi)^2 + \sin^2 m\pi \text{ sh}^2\, m\Theta^k} \frac{1}{\text{ch } m\Theta^k + \sin m\varphi}, \tag{108}$$

where

$$\Theta^k \equiv \Theta^k_-.$$

From (92) and (96) it is readily seen that

$$\lim_{\Theta_+^l \to \infty} \sigma_{\text{rearr}}(-l^2) = \lim_{\Theta_+^k \to \infty} \sigma_{\text{rearr}}(k^2) = 0.$$

In order to verify the flux conservation condition for $E > 0$ we use the following result which is derived by changing the variables in the integral written below in such a way as to obtain an integration from $-\infty$ to ∞ of a rational function to which the method of residues can be applied.

Thus:

$$\int_{-\Phi}^{\Phi} \frac{\cos^2 m\varphi}{[(\cos m\pi \operatorname{ch} m\Theta^k + \sin m\varphi)^2 + \sin^2 m\pi \operatorname{sh}^2 m\Theta^k] (\operatorname{ch} m\Theta^k + \sin m\varphi)} d\varphi =$$

$$= \frac{\pi}{2m} \frac{1}{\operatorname{sh} m\Theta^k (\operatorname{sh}^2 m\Theta^k + \sin^2 m\pi/2)}. \tag{109}$$

Employing (109) we find

$$\int_{-\Phi}^{\Phi} \bar{\sigma}_{\text{diss}}(k^2, \varphi) d\varphi \equiv \bar{\sigma}_{\text{diss}}(k^2) = \frac{\sin^2 m\pi/2}{\operatorname{sh}^2 m\Theta^k + \sin^2 m\pi/2} = 1 - \bar{\sigma}_{\text{el}}(k^2) \tag{110}$$

in agreement with the unitarity condition above the break-up threshold for this model and also consistently with the result, which can be obtained by putting $\Theta_+^k \to \infty$ in (98).

It is convenient to express the cross-sections as functions of the parameter

$$\lambda = \frac{\sqrt{E+\alpha^2}}{-\alpha}, \quad \text{where} \quad \alpha \equiv \alpha_3. \tag{111}$$

λ is a square root of the ratio of the incident energy of particle 3 to the binding energy of the pair (1,2).

We have

$$0 < \lambda < 1 \quad \text{for } -\alpha^2 < E < 0 \quad \text{and } \lambda > 1 \quad \text{for } E > 0; \tag{112}$$

$$\Theta^k = \frac{1}{2} \ln \frac{\lambda+1}{\lambda-1}, \tag{113}$$

$$\operatorname{sh} m\Theta^k = \frac{1}{2}\left[\frac{(\lambda+1)^m - (\lambda-1)^m}{(\lambda^2-1)^{m/2}}\right], \tag{114}$$

$$\bar{\sigma}_{\text{diss}}(\lambda) = 1 - \bar{\sigma}_{\text{el}}(\lambda) = 4 \sin^2 m\pi/2 \left[\left(\frac{\lambda+1}{\lambda-1}\right)^m - 2 + \left(\frac{\lambda-1}{\lambda+1}\right)^m + 4 \sin^2 m\pi/2\right]^{-1}. \tag{115}$$

When substituting $m = 4/3$ into (115) we obtain

$$\bar{\sigma}_{\text{diss}}(\lambda) = 3\left(\frac{\lambda-1}{\lambda+1}\right)^{4/3}\left[1+\left(\frac{\lambda-1}{\lambda+1}\right)^{4/3}+\left(\frac{\lambda-1}{\lambda+1}\right)^{8/3}\right]^{-1} \tag{116}$$

in agreement with the result obtained by Nussenzweig in Ref. [6].

We note that $\bar{\sigma}_{\text{diss}}(\lambda)$ is independent of the coordinate system used. Employing (115) we find the behaviour of the elastic and break-up cross-sections for large values of the incident energy:

$$\bar{\sigma}_{\text{el}}(\lambda) \underset{\lambda\to\infty}{\cong} \frac{m^2}{\lambda^2}\cdot\frac{1}{\sin^2 m\pi/2}, \tag{117}$$

$$\bar{\sigma}_{\text{diss}}(\lambda) \underset{\lambda\to\infty}{\to} 1. \tag{118}$$

For $m = 4/3$ we have

$$\bar{\sigma}_{\text{el}}(\lambda) \cong \tfrac{64}{9}\,\lambda^{-2},$$

a result obtained by Nussenzweig.

The behaviour of the break-up cross-section for values of λ in the vicinity of the break-up threshold is as follows:

$$\sigma_{\text{diss}}(\lambda) \underset{\lambda\to1^+}{\cong} 4\sin^2 m\pi/2\left(\frac{\lambda-1}{2}\right)^m. \tag{119}$$

The total break-up cross-section considered as funcrtion of m for given Θ^k has zeros for $m = 2n$, where n is an integer; and maxima for m satisfying the equation

$$\pi/2 \cos m\pi/2 \operatorname{sh} m\Theta^k = \Theta^k \sin m\pi/2 \operatorname{ch} m\Theta^k. \tag{120}$$

The height of the maxima decreases for increasing m.

8. Concluding remarks

The number of the three-body models which can be solved exactly is very limited. The review of the one-dimensional problems for which the analytic solutions in the configuration space were found is given in Ref. [6]. The model considered here distinguishes itself from this group of models in that the mass ratios and the strengths of the two-body interactions are arbitrary. A full discussion of the results presented here will be interesting both for illustrative purposes and for the examinations of the models with more realistic potentials. It would be also interesting to study parallelly the Faddeev equations for the system described above.

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APPENDIX

Here we collect some properties of the special functions $M_\Phi(z)$ introduced by Malyuzhinetz [2], [3]. In the quoted literature this special function is denoted by $\Psi_\Phi(z)$.

The definition and integral representations:

$$M_\Phi(z) = \prod_{n=1}^{\infty} \prod_{m=1}^{\infty} \left[1 - \left(\frac{z}{2\Phi(2n-1)+\pi/2(2m-1)}\right)^2\right]^{(-1)^{m+1}}, \tag{A.1}$$

$$M_\Phi(z) = \exp\left[\frac{i}{8\Phi}\int_0^z d\mu \int_{-i\infty}^{i\infty} \operatorname{tg}\frac{\pi v}{4\Phi}\cdot\frac{dv}{\cos(v-\mu)}\right], \tag{A.2}$$

$$M_\Phi(z) = \exp\left[-\frac{1}{2}\int_0^{\infty} \frac{\operatorname{ch} zs - 1}{s \operatorname{ch}\pi/2s \cdot \operatorname{sh} 2\Phi s}\right] ds, \tag{A.3}$$

where the last integral defines $M_\Phi(z)$ in the strip $|\operatorname{Re} z| < 2\Phi+\pi/2$. For z outside this strip we make use of the relations (A.8)–(A.9) (see below). From the expressions (A.1)–(A.3) we see immediately that

$$M_\Phi(z) = M_\Phi(-z). \tag{A.4}$$

The zeros and poles nearest to the point $z = 0$ are

$$\left.\begin{aligned} z &= \pm(\pi/2+2\Phi) \\ z &= \pm(3\pi/2+2\Phi) \end{aligned}\right\} \Phi > 0 \tag{A.5}$$

respectively.

For the values of Φ such that $4\Phi/\pi = n/m$, where n and m are integers, and the fraction n/m is irreducible, we have

$$M_\Phi(z) = \prod_{k=1}^{m}\prod_{l=1}^{n}\left(\frac{\cos 1/2a(k,l)}{\cos 1/2[z/n+a(k,l)]}\right)^{(-1)^l} \quad \text{for } n \text{ odd,}$$

$$M_\Phi(z) = \prod_{k=1}^{m}\prod_{l=1}^{n}\exp\left[\frac{(-1)^l}{\pi}\int_{a(k,l)}^{a(k,l)+z/n} u \operatorname{ctg} u\, du\right] \quad \text{for } n \text{ even,} \tag{A.6}$$

where

$$a(k, l) = \pi/2\left(\frac{2l-1}{n} - \frac{2k-1}{m}\right).$$

The function $M_\Phi(z)$ may be expressed in terms of the Barnes' unsymmetric double gamma function $G(\alpha, \beta)$ in the following way [17]

$$M_\Phi(z) = C\pi^{\pi/4\Phi}\,\frac{G(\pi/8\Phi+1/2-z/4\Phi, \pi/2\Phi)G(\pi/8\Phi+1/2+z/4\Phi, \pi/2\Phi)}{G(3\pi/8\Phi+1/2+z/4\Phi, \pi/2\Phi)G(3\pi/8\Phi+1/2-z/4\Phi, \pi/2\Phi)}, \tag{A.7}$$

where C is a constant.

The solution of the wedge diffraction problem in terms of the functions $G(\alpha, \beta)$ hat been given by Williams [8].

It should be noted that the closed form of the Green function for the problem (7) was also found and expressed in terms of the functions $M_\Phi(z)$ [18]. The tables of $M_\Phi(z)$ were constructed by Zavadskij and Sakharova [4].

The most important properties of $M_\Phi(z)$ are:

$$\frac{M_\Phi(z+2\Phi)}{M_\Phi(z-2\Phi)} = \operatorname{ctg} 1/2(z+\pi/2), \tag{A.8}$$

$$M_\Phi(z+\pi/2)M_\Phi(z-\pi/2) = [M_\Phi(\pi/2)]^2 \cos \pi z/4\Phi, \tag{A.9}$$

$$M_\Phi(z+\Phi)M_\Phi(z-\Phi) = [M_\Phi(\Phi)]^2 M_{\Phi/2}(z). \tag{A.10}$$

The bahaviour of $M_\Phi(z)$ for large values of $|\operatorname{Im} z|$ is as follows [4]:

$$M_\Phi(z) \underset{|\operatorname{Im} z|\to\infty}{\cong} \left(\cos\frac{\pi z}{4\Phi}\right)^{1/2} C_\Phi, \tag{A.11}$$

where

$$C_\Phi = \exp\left\{-1/2\pi \int_{-\infty}^{\infty} \frac{\ln \operatorname{ch} \frac{\pi s}{4\Phi}}{\operatorname{ch} s}\, ds\right\}.$$

Using the definition (A.3) we find, that

$$\frac{M_\Phi(2\Phi-\pi/2)}{M_\Phi(\pi/2)} = \sqrt{\frac{\pi}{2\Phi}}, \quad \Phi > 0. \tag{A.12}$$

In the calculations of the cross-sections we have to know the properties of the moduli of $M_\Phi(z)$. From the definitions of $M_\Phi(z)$ we see that

$$|M_\Phi(\alpha+i\beta)| = |M_\Phi(\alpha-i\beta)| \quad \alpha, \beta \text{ real.} \tag{A.13}$$

Employing (A.13) and the functional relations (A.8) and (A.9) we find that

$$\left|\frac{M_\Phi(\pi/2+i\alpha)}{M_\Phi(\pi/2)}\right|^2 = \operatorname{ch}\frac{\pi\alpha}{4\Phi}, \tag{A.14}$$

$$\left|\frac{M_\Phi(2\Phi+\pi/2+i\alpha)}{M_\Phi(\pi/2)}\right|^2 = \operatorname{sh}\frac{\pi\alpha}{4\Phi}\cdot \operatorname{th}\frac{\alpha}{2}. \tag{A.15}$$

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