EFFECTIVE MAGNON HAMILTONIAN FOR ITINERANT-ELECTRON FERROMAGNETICS: II. APPLICATIONS TO MAGNON INTERACTION PROBLEMS

BY J. MORKOWSKI, Z. KRÓL AND S. KROMPIEWSKI

Ferromagnetics Laboratory, Institute of Physics of the Polish Academy of Sciences, Poznań*

(Received November 29, 1972)

The method of the effective magnon Hamiltonian, formulated in the preceding paper, is applied to the discussion of some problems involving magnon interaction in the itinerant electron model of ferromagnetism. The model of a single narrow band described by the Hubbard Hamiltonian implemented by the energy of dipolar interactions of itinerant electrons is considered. First, the four-magnon relaxation processes are discussed. Second, the influence of magnon interaction on the low-temperature spontaneous magnetization is estimated.

1. Introduction

In the preceding paper [1] (referred to as I hereafter) a method of an effective magnon Hamiltonian was described. The method is particularly well suited for studying magnon relaxation processes, since it enables us to use the well-known calculation procedures developed for localized-spin ferromagnetics.

The effective Hamiltonian allows us to discuss separately different relaxation mechanisms and to find the resultant inverse relaxation time as the sum of inverse relaxation times for different processes. In [2] the relaxation time for 3-magnon confluence processes was calculated. In the present paper we shall discuss 4-magnon relaxation due to Coulomb and magnetic dipolar interaction between magnetic electrons.

2. Hamiltonian

We shall consider the case of strong magnetic field H, *i. e.*, $H \gg 4\pi M$ where M is the magnetization. In this limit the terms $L_q\beta_q\beta_{-q}+$ h. c. in the effective Hamiltonian (I.16) (the formula (16) of I is here referred to as (I.16), *etc.*) can be neglected and the energy of long wavelengths magnons $\tilde{E}_q$ simplifies to $\tilde{E}_q = K_q = E_q$ where E_q is given by (I.21). The

* Address: Zakład Fizyki Ferromagnetyków, Instytut Fizyki PAN, Noskowskiego 10, 61-704 Poznań, Poland.

terms like $C_{qq'}\beta^{+}_{q+q'}\beta_q\beta_{q'}$ in the total effective Hamiltonian (I.16) are irrelevant for our present problem. Therefore, the effective Hamiltonian for 4-magnon relaxation processes is (the notation is the same as in I)

$$\mathscr{H} = \sum_q E_q\beta_q^+\beta_q + \sum_{kk'q} G^q_{kk'}\beta^+_{k+q}\beta^+_{k'-q}\beta_k\beta_{k'} + \\ + \sum_{qq'q''} (F_{qq'q''}\beta^+_{q+q'+q''}\beta_q\beta_{q'}\beta_{q''} + \text{h.c.}), \tag{1}$$

where the magnon energy E_q is calculated from the equation (I.3)

$$IN^{-1}\sum_k n_k(\varepsilon_{k+q} - \varepsilon_k + \Delta - E_q)^{-1} = 1, \tag{2}$$

and

$$G^q_{kk'} = \tfrac{1}{4}\langle[\beta_{k+q}, [\beta_{k'-q}, [[\mathscr{H}, \beta^+_{k'}], \beta^+_k]]]\rangle, \tag{3}$$

$$F_{qq'q''} = \tfrac{1}{6}\langle[\beta_{q+q'+q''}, [[[\mathscr{H}, \beta^+_{q''}], \beta^+_{q'}], \beta^+_q]]\rangle. \tag{4}$$

Here $\langle\ldots\rangle$ denotes the average over the ground state and $\mathscr{H}$ is the Hamiltonian (I.13) of the system of itinerant magnetic electrons, including the Coulomb and magnetic dipolar interactions. We assume that in the ground state the spin-down band (spin-down electrons have their magnetic moments parallel to the magnetic field direction) is partially occupied whereas the spin-up band is empty, *i. e.*, we consider the case of a strong itinerant ferromagnet. n_k denotes the ground state occupation number for spin-down electrons.

Simple calculations lead to the following expressions for the coefficients of the 4-magnon terms:

$$G^q_{kk'} = \tfrac{1}{4}(\bar{G}^q_{kk'} + \bar{G}^{-q}_{k'k} + \bar{G}^{k'-k-q}_{kk'} + \bar{G}^{-k'+k+q}_{k'k}) \tag{5a}$$

where

$$\begin{aligned}\bar{G}^q_{kk'} = & -IN^{-1}\sum_{pp'} b_{p+k',p}b_{p'+k+q,p'+q}b^*_{p+k'-q,p}b^*_{p'+k+q,p'}n_pn_{p'} + \\ & +IN^{-1}\sum_{pp'} b_{p+k',p}b_{p+k+q,p+q}b^*_{p'+k'-q,p'}b^*_{p+k+q,p}n_pn_{p'} + \\ & +\tfrac{1}{2}D_qN^{-1}\sum_{pp'} b_{p+k',p}b_{p'+k+q,p'+q}b^*_{p+k'-q,p}b^*_{p'+k+q,p'}n_pn_{p'} + \\ & +\tfrac{1}{4}D_qN^{-1}\sum_{pp'} b_{p+k',p}b_{p'+k,p'}b^*_{p+k'-q,p}b^*_{p'+k+q,p'}n_pn_{p'} + \\ & +\tfrac{1}{4}D_qN^{-1}\sum_{pp'} b_{p'+k'-q,p'-q}b_{p+k+q,p+q}b^*_{p'+k'-q,p'}b^*_{p+k+q,p}n_pn_{p'} + \\ & +\tfrac{1}{2}D_{k'-q}N^{-1}\sum_{pp'} b_{p'+k',p'}b_{p'+k+q,p'+q}b^*_{p+k'-q,p}b^*_{p'+k+q,p'}n_pn_{p'} - \\ & -\tfrac{1}{2}N^{-1}\sum_{pp'} D_{p'-p-q}b_{p+k',p}b_{p+k+q,p+q}b^*_{p'-q+k',p'}b^*_{p+k+q,p}n_pn_{p'} - \\ & -\tfrac{1}{4}N^{-1}\sum_{pp'} D_{p'-p-q}b_{p+k',p}b_{p'+k,p'}b^*_{p'-q+k',p'}b^*_{p+k+q,p}n_pn_{p'} -\end{aligned}$$

$$-\tfrac{1}{4}N^{-1}\sum_{pp'} D_{p'-p+q}b_{p+k'-q,p-q}b_{p'+q+k,p'+q}b^*_{p+k'-q,p}b^*_{p'+q+k,p'}n_pn_{p'}-$$

$$-\tfrac{1}{2}N^{-1}\sum_{pp'} D_{p'-p-q+k'}b_{p'+k',p'}b_{p+q+k,p+q}b^*_{p'-q+k',p'}b^*_{p+k+q,p}n_pn_{p'}, \tag{5b}$$

and

$$F_{qq'q''} = \tfrac{1}{6}(\bar{F}_{qq'q''}+\bar{F}_{q''qq'}+\bar{F}_{q'q''q}+\bar{F}_{q''q'q}+\bar{F}_{qq''q'}+\bar{F}_{q'qq''}) \tag{6a}$$

where

$$\bar{F}_{qq'q''} = B^*_qN^{-1}\sum_{kk'} b_{k+q+q'+q'',k+q+q'}b_{k+q',k}b_{k'+q,k'}b^*_{k+q+q'+q'',k}n_kn_{k'}+$$

$$+B^*_{q'}N^{-1}\sum_{kk'} b_{k'+q+q'+q'',k'+q+q'}b_{k+q',k}b_{k'+q,k'}b^*_{k'+q+q'+q'',k'}n_kn_{k'}-$$

$$-N^{-1}\sum_{kk'} B^*_{k'-k-q}b_{k+q+q'+q'',k+q+q'}b_{k'+q',k'}b_{k+q,k}b^*_{k+q+q'+q'',k}n_kn_{k'}-$$

$$-N^{-1}\sum_{kk'} B^*_{k'-k+q'}b_{k'+q+q'+q'',k'+q+q'}b_{k'+q',k'}b_{k+q,k}b^*_{k'+q+q'+q'',k'}n_kn_{k'}. \tag{6b}$$

All the symbols used here are defined in I.

The two-body interaction term in the effective Hamiltonian (1) proportional to the matrix elements $G^q_{kk'}$ is determined by electrostatic as well as magnetic dipolar interactions, whereas the one-in and three-out interaction term is of purely dipolar origin. It is convenient to divide $\bar{G}^q_{kk'}$ into the electrostatic $\bar{\Gamma}^q_{kk'}$ and dipolar $P^q_{kk'}$ contributions

$$\bar{G}^q_{kk'} = \bar{\Gamma}^q_{kk'}+P^q_{kk'}. \tag{7}$$

$\bar{\Gamma}^q_{kk'}$ contains those terms in (5b) which are proportional to the Coulomb integral I, *cf.* Eq. (I.12); the dipolar terms belong to $P^q_{kk'}$.

Calculation of the coefficients (5), (6) of the effective Hamiltonian for a realistic band structure would be an extremely tedious task. We confine ourselves to the case of parabolic shape of the band taking for the electron energy $\varepsilon_k = \hbar^2k^2/2m^*$.

The coefficients $G^q_{kk'}$ and $F_{qq'q''}$ were calculated as expansions in powers of wave vectors. The electrostatic contribution $\Gamma^q_{kk'}$ was calculated up to the terms of the fourth order with respect to the wave vectors $\vec{q}$, $\vec{k}$, $\vec{k}'$. The dipolar terms $P^q_{kk'}$ and $F_{qq'q''}$, which are much smaller, were calculated in lowest order with respect to the wave vectors and in this approximation they depend only on the wave vector directions.

It is covenient to expand $b_{k+q,k}$, given by Eqs (I.5) and (I.6), in the following form:

$$b_{k+q,k} \cong b_{q,0}[1-\chi_q\vec{q}\cdot\vec{k}+\chi_q^2(\vec{q}\cdot\vec{k})^2-$$

$$-\chi_q^3(\vec{q}\cdot\vec{k})^3+\chi_q^4(\vec{q}\cdot\vec{k})^4+\ldots] \tag{8}$$

where

$$b_{q,0} \cong (nN)^{-1/2}(1-\tfrac{3}{10}\chi_q^2k_F^2q^2+\ldots) \tag{9}$$

and

$$\chi_q = \hbar^2\Big/\left[m^*\left(\Delta-E_q+\frac{\hbar^2}{2m^*}q^2\right)\right]. \tag{10}$$

Here, n denotes the number of electrons per atom; N is the number of atoms; k_F is the Fermi momentum and $\varepsilon_F = \hbar^2 k_F^2/2m^*$ is the Fermi energy for spin-down electrons; $\Delta = 2\mu_B H + nI$ is the energy needed to reverse the electron spin from down to up direction; $E_q = 2\mu_B H + \alpha q^2 + \ldots$ is the magnon energy, which appears as the solution of equation (I.3). The magnon energy parameter α could be expressed in terms of Δ, k_F and m^*. However, the Random Phase Approximation taken as the basis for our treatment is insufficient for accurate calculations of α; also, the effective mass approximation is too crude for quantitative results (see *e. g.* [3]). It is therefore reasonable not to determine α from approximate treatments but, on the contrary, to consider α as an independent parameter which can finally be estimated from experimental data.

Straightforward though laborious calculations yield the result, correct up to the terms of the fourth order with respect to the wave vectors,

$$\begin{aligned} \Gamma^q_{kk'} = \tfrac{4}{5} I N^{-1} \eta^2 k_F^{-2} \Big\{ & \vec{k}\cdot\vec{k}' + \\ & + \chi_0 \Big[\frac{1}{2}\Big(\frac{\alpha k_F^2}{\varepsilon_F} - 1 + \frac{36}{35}\eta \Big)(k^2 + k'^2)(\vec{k}\cdot\vec{k}') + \\ & + (1 - \tfrac{17}{35}\eta)(k'^2(\vec{k}\cdot\vec{q}) - k^2(\vec{k}'\cdot\vec{q}) - 2(\vec{k}\cdot\vec{q})(\vec{k}'\cdot\vec{q})) + \\ & + (1 + \tfrac{1}{35}\eta)(\vec{k}'\cdot\vec{q} - \vec{k}\cdot\vec{q})(\vec{k}\cdot\vec{k}') - \tfrac{18}{35}\eta q^2 (\vec{k}\cdot\vec{k}') + \\ & + \tfrac{1}{2}(1 - \tfrac{6}{7}\eta) k^2 k'^2 - \tfrac{1}{2}(1 - \tfrac{66}{35}\eta)(\vec{k}\cdot\vec{k}')^2 \Big] \Big\} \end{aligned} \tag{11}$$

where

$$\chi_0 = \hbar^2/[m^*(\Delta - 2\mu_B H)] \cong \hbar^2/(m^*\Delta) \text{ and } \eta = \varepsilon_F/\Delta.$$

The dipolar contribution to $\overline{G}^q_{kk'}$ is in lowest order

$$P^q_{kk'} = -\tfrac{1}{2} C(4d_q + 4d_{k'-k-q} + d_{k+q} + d_{k'-q} + d_k + d_{k'}) \tag{12}$$

where

$$d_q = 1 - 3(q_z/q)^2 \tag{13}$$

and

$$C = (2\pi/3)\mu_B^2/V. \tag{14}$$

Similarly,

$$F_{qq'q''} = -C(f_q + f_{q'} + f_{q''}) \tag{15}$$

with

$$f_q = (q_x - iq_y)^2/q^2. \tag{16}$$

It is interesting to note that the dipolar part of the effective magnon Hamiltonian (1) in the limit of long magnon wavelengths appears to be the same as the corresponding part of the magnon Hamiltonian for the localized-electrons ferromagnets.

We shall illustrate applications of the effective Hamiltonian considering, first, four-magnon relaxation and, second, the effect of magnon interaction on the low-temperature spontaneous magnetization.

3. *Four-magnon relaxation*

The magnon relaxation time can be calculated from a linearized kinetic equation by the procedure used long ago by Akhiezer [4] (see [5] for a modern description).

The four-magnon scattering processes as defined by the effective Hamiltonian (1) consist of two-in and two-out processes, determined by the terms proportional to $G^{q}_{kk'}$, and one-in and three-out processes determined by $F_{qq'q''}$. It is convenient to describe the system in terms of eigenstates of the magnon number operators $\beta^{+}_{q}\beta_{q}$; let N_q denote their eigenvalues or magnon occupation numbers.

The transition probability per unit time, as calculated from the golden rule of the first-order perturbation theory, is for two-in and two-out processes

$$W[(N_{k+q}, N_{k'-q}, N_k, N_{k'}) \to (N_{k+q}+1, N_{k'-q}+1, N_k-1, N_{k'}-1)] = \\ = (2\pi/\hbar)\, |4G^{q}_{kk'}|^2 (N_{k+q}+1)\,(N_{k'-q}+1) N_k N_{k'} \times \\ \times \delta(E_{k+q}+E_{k'-q}-E_k-E_{k'}). \tag{17}$$

The transition probability per unit time for one-in and three-out processes is

$$W[(N_q, N_{q'}, N_{q''}, N_{q+q'+q''}) \to (N_q-1, N_{q'}-1, N_{q''}-1, N_{q+q'+q''}+1)] = \\ = (2\pi/\hbar)\, |6F_{qq'q''}|^2 N_q N_{q'} N_{q''} (N_{q+q'+q''}+1) \times \\ \times \delta(E_q+E_{q'}+E_{q''}-E_{q+q'+q''}). \tag{18}$$

The net rate of change of the magnon occupation number N_q is given by the kinetic equation

$$\frac{dN_q}{dt} = \tfrac{1}{2}\sum_{kk'} (2\pi/\hbar)\, |4G^{q-k}_{kk'}|^2 \{(N_q+1)\,(N_{k'+k-q}+1) N_k N_{k'} - \\ - N_q N_{k'+k+q} (N_k+1)\,(N_{k'}+1)\} \delta(E_q+E_{k'+k-q}-E_k-E_{k'}) + \\ + \tfrac{1}{2}\sum_{kk'} (2\pi/\hbar) |6F_{qk'k''}|^2 \{(N_q+1)\,(N_k+1)\,(N_{k'}+1) N_{q+k+k'} - \\ - N_q N_k N_{k'} (N_{q+k+k'}+1)\} \delta(E_q+E_k+E_{k'}-E_{q+k+k'}) + \\ + \tfrac{1}{2}\sum_{kk'} (2\pi/\hbar)\, |6F_{k,k',q-k-k'}|^2 \{(N_q+1) N_k N_{k'} N_{q-k-k'} - \\ - N_q (N_k+1)\,(N_{k'}+1)\,(N_{q-k-k'}+1)\} \delta(E_k+E_{k'}+E_{q-k-k'}-E_q). \tag{19}$$

Let $\overline{N}_k = [\exp(\beta E_k)-1]^{-1}$ be the equilibrium value of N_k ($\beta = 1/k_B T$) and let the magnon system be not far from equilibrium, *i. e.*, let $N_k - \overline{N}_k$ be small. Then we can linearize the right-hand side of the kinetic equation (19) with respect to deviations from equilibrium, $N_k - \overline{N}_k$. The linearized kinetic equation takes the form

$$\frac{dN_q}{dt} = -\frac{1}{\tau_q}(N_q - \overline{N}_q) - \sum_{k \neq q}{}' (\ldots)(N_k - \overline{N}_k). \tag{20}$$

The inverse relaxation $1/\tau_q$ for magnons of wave vector $\vec{q}$ can be conveniently divided into the contribution from the two-in and two-out processes, $1/\tau_q^{(2,2)}$, and the contribution $1/\tau_q^{(1,3)}$ from the one-in and three-out processes,

$$1/\tau_q = 1/\tau_q^{(2,2)} + 1/\tau_q^{(1,3)}. \tag{21}$$

These two contributions, as derived from (19), are

$$1/\tau_q^{(2,2)} = \sum_{kk'} (16\pi/\hbar)\, |G_{kk'}^{q-k}|^2 \{\overline{N}_{k+k'-q}(\overline{N}_k + \overline{N}_{k'} + 1) - \overline{N}_k \overline{N}_{k'}\} \delta(E_q + E_{k+k'-q} - E_k - E_{k'}), \tag{22}$$

$$1/\tau_q^{(1,3)} = 1/\tau_q^{c(1,3)} + 1/\tau_q^{s(1,3)}. \tag{23a}$$

The first component of Eq. (23a) comes from 4-magnon confluence processes,

$$1/\tau_q^{c(1,3)} = \sum_{kk'} (36\pi/\hbar)\, |F_{qkk'}|^2 \{\overline{N}_k \overline{N}_{k'} - (\overline{N}_k + \overline{N}_{k'} + 1)\overline{N}_{q+k+k'}\} \delta(E_q + E_k + E_{k'} - E_{q+k+k'}), \tag{23b}$$

and the second from 4-magnon splitting processes

$$1/\tau_q^{s(1,3)} = \sum_{kk'} (36\pi/\hbar)\, |F_{k,k',q-k-k'}|^2 \{(\overline{N}_k + 1)(\overline{N}_{k'} + 1) \times (\overline{N}_{q-k-k'} + 1) - \overline{N}_k \overline{N}_{k'} \overline{N}_{q-k-k'}\} \delta(E_k + E_{k'} + E_{q-k-k'} - E_q). \tag{23c}$$

$1/\tau_q^{(1,3)}$ is determined by dipolar interaction alone, whereas $1/\tau_q^{(2,2)}$ is due to electrostatic and dipolar interactions. For very small q dipolar contributions to the inverse relaxation time dominate over the exchange ones. In the limit $q = 0$, $1/\tau_0$ is purely of dipolar origin. However, for large enough q the scattering processes due to electrostatic interaction are much more efficient than the dipolar ones and $1/\tau_q$ is then practically determined by the electrostatic terms.

For simplicity we consider the case of strong magnetic field, $H \gg 4\pi M$. Consequently, we can neglect dipolar corrections to the magnon energy and take for long-wavelengths magnons the simple formula $E_q = 2\mu_B H + \alpha q^2$ for the magnon energy.

a) Relaxation time for $q = 0$

For $q = 0$ the energy conservation condition expressed by the Dirac δ-function in (22) reduces to $\delta(E_0 + E_{k+k'} - E_k - E_{k'}) = (1/2\alpha)\,\delta(\vec{k} \cdot \vec{k}')$. The Coulomb contribution $(\Gamma_{kk'}^{q-k})_{q=0}$ to $(G_{kk'}^{q-k})_{q=0}$ is, in the first approximation with respect to the magnon wave vectors, proportional to $(\vec{k} \cdot \vec{k}')$, as is seen from Eq. (11). Therefore, in the first approximation $\Gamma_{kk'}^{-k}\delta(\vec{k}\cdot\vec{k}')$ vanishes and the relaxation of the uniform magnons $q = 0$ is determined solely by dipolar interactions.

Here we encounter a difficulty of the method of the effective magnon Hamiltonian. From general considerations, namely, from the fact that the Hubbard Hamiltonian commutes with the total magnetic moment of the system of itinerant electrons we expect that the uniform magnon should not be damped by interactions originating from the Coulomb part of the Hubbard Hamiltonian (I.1). The effective Hamiltonian with $\Gamma_{kk'}^{q}$ given by Eq. (11) preserves this property only in the first approximation. Some of the terms in Eq. (11) of the fourth order with respect to the wave vectors would lead to a spurious contribution to the inverse relaxation time of uniform magnons. Fortunately, this spurious contribution rapidly diminishes with decreasing temperature, roughly like the fifth power of temperature, and at sufficiently low temperatures becomes negligible as compared with the dipolar contribution.

Standard calculations give simple results for the two components $1/\tau_0^{(2,2)}$ and $1/\tau_0^{(1,3)}$ of $1/\tau_0$ (for simplicity we assume that the sample is an ellipsoid of revolution with the symmetry axis along the magnetic field direction):

$$1/\tau_0^{(2,2)} = \frac{1}{36\pi}\left(\frac{124}{5}+d^2\right)\mu_B^4\hbar^{-1}\alpha^{-3}(k_BT)^2 f(\varepsilon) \tag{24}$$

where $d = 1-3N_z$ (N_z is the demagnetizing factor) and

$$f(\varepsilon) = \int_0^\infty dx \int_0^\infty dy\{(e^{\varepsilon+x+y}-1)^{-1}[(e^{\varepsilon+x}-1)^{-1}+ +(e^{\varepsilon+y}-1)^{-1}+1]-(e^{\varepsilon+x}-1)^{-1}(e^{\varepsilon+y}-1)^{-1}\} \tag{25}$$

with $\varepsilon = 2\mu_B H/k_B T$.

For $q = 0$ the 4-magnon splitting processes are forbidden as they fail to fulfill the energy conservation condition. Consequently, $1/\tau_0^{s(1,3)} = 0$. $1/\tau_0^{(1,3)}$ is determined by 4-magnon confluence processes and is given by the expression

$$1/\tau_0^{(1,3)} = 1/\tau_0^{c(1,3)} = \frac{2}{15\pi}\mu_B^4\hbar^{-1}\alpha^{-3}(k_BT)^2 g(\varepsilon) \tag{26}$$

where

$$g(\varepsilon) = \int_0^\infty dx \int_0^\infty dy\,\theta(xy-\varepsilon^2)\left(1+3\frac{\varepsilon^2}{xy}\right)\left\{\frac{1}{e^{\varepsilon+x}-1}\,\frac{1}{e^{\varepsilon+y}-1} - \left(\frac{1}{e^{\varepsilon+x}-1}+\frac{1}{e^{\varepsilon+y}-1}+1\right)\frac{1}{e^{3\varepsilon+x+y}-1}\right\}. \tag{27}$$

$\theta(x)$ is equal to 1 for positive x and vanishes for negative x.

Calculation of $f(\varepsilon)$ is simple. It can be represented by the expansion

$$f(\varepsilon) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\varepsilon}. \tag{28a}$$

Except at extremely low temperatures, for values of the magnetic field strengths within the experimentally accessible range $\varepsilon = 2\mu_B H/k_B T$ is very small as compared with unity. For $\varepsilon \ll 1$ we can replace (28a) by a more convenient expansion

$$f(\varepsilon) = \frac{\pi^2}{6} + \varepsilon \ln(1-e^{-\varepsilon}) - \sum_{n=1}^{\infty} \frac{1}{n^2}(1-e^{-\varepsilon})^n. \tag{28b}$$

Calculation of $g(\varepsilon)$ is cumbersome. In a crude approximation we have for $\varepsilon = 0$:

$$g(0) \cong 5.4. \tag{29}$$

b) Relaxation time for finite q

Magnons having finite wave-vector q relax by the combined effect of dipolar and electrostatic interactions. In order to find the relaxation time for magnons of very small wave-vector q we have to add to $1/\tau_0$, calculated in the preceding subsection, the contribution from 3-magnon confluence processes discussed in [2]. As the wave-vector q increases, all the dipolar processes: 3-magnon confluence and splitting, 4-magnon confluence and splitting, 4-magnon scattering processes have to be taken into account, as well as the 4-magnon scattering processes due to electrostatic interaction. For large enough q, the latter processes dominate. Here we shall calculate the inverse relaxation time determined by the 4-magnon scattering processes.

We use the general formula (22) and in the coefficient $G_{kk'}^{q-k}$ we neglect the dipolar contribution leaving only the electrostatic part $\Gamma_{kk'}^{q-k}$, *cf.* Eq. (7). Moreover, in the first approximation we retain only the leading term in the expansion of $\Gamma_{kk'}^{q-k}$, *i. e.*, we take $\Gamma_{kk'}^{q} = (4/5)\,\eta^2\, IN^{-1}\, k_F^{-2}\, \vec{k}\cdot\vec{k}'$. After standard calculations we get for the inverse relaxation time of magnons q (averaged over the directions of $\vec{q}$) as determined by the two-two scattering processes due to the electrostatic interaction:

$$\langle 1/\tau_q^{(2,2)e}\rangle = (6\pi/125)\,\frac{n^2 I^2}{\hbar\alpha k_F^2}\,\eta^4(\beta\alpha k_F^2)^{-4}\times$$

$$\times\frac{1}{p}\int_0^{\infty} dx \int_0^{\infty} dy\, \psi(x, y)\varphi(x, y)\theta(x+y-p^2). \tag{30}$$

The notation is as follows:

$$\psi(x, y) = \frac{1}{e^{\varepsilon+x+y-p^2}-1}\left(\frac{1}{e^{\varepsilon+x}-1} + \frac{1}{e^{\varepsilon+y}-1} + 1\right) -$$

$$-\frac{1}{e^{\varepsilon+x}-1}\,\frac{1}{e^{\varepsilon+y}-1}, \tag{31}$$

$$\varphi(x, y) = (x+y)^{5/2}\{(3\mu^2+8)(\sqrt{1+\mu}-\sqrt{1-\mu})-4\mu(\sqrt{1+\mu}+\sqrt{1-\mu})\}, \tag{32a}$$

$$\mu = \min\left(\frac{2\sqrt{xy}}{x+y},\ \frac{2p}{x+y}\sqrt{x+y-p^2}\right), \tag{32b}$$

and

$$p = q\sqrt{\beta\alpha}, \qquad \varepsilon = 2\mu_B H\beta.$$

(For reasonably low temperatures the only singnificant contributions to the integral in Eq. (30) come from small values of x, y, corresponding to the central region of the Brillouin zone. Therefore as usually, we replace the upper limits by infinity instead of the values defined by the Brillouin zone boundary. The same remark applied also to the integrals in Eqs (52) and (27).)

From Eq. (30) it is easy to calculate the relaxation time for long wavelengths magnons and for not too low temperatures, within the condition $\alpha q^2 \ll k_B T$. In the absence of the external magnetic field ($\varepsilon = 0$) in lowest order with respect to $(\alpha q^2/k_B T)$ we have from Eqs (30)–(32):

$$\frac{1}{\tau_q^{(2,2)e}} = \frac{64\pi}{25}\,\frac{n^2 I^2}{\hbar\alpha k_F^2}\,\eta^4 \left(\frac{k_B T}{\alpha k_F^2}\right)^2 (q/k_F)^4 \times$$

$$\times \left\{\ln^2\left(\frac{\alpha q^2}{k_B T}\right) - C_1 \ln\left(\frac{\alpha q^2}{k_B T}\right) + C_2 + O(q^2)\right\}. \tag{33}$$

C_1 and C_2 are numerical coefficients; their approximate values are $C_1 = 0.9$, $C_2 = 2.6$.

In the limit of very low temperatures the magnon occupation numbers $\overline{N}_k$ are small and we can neglect in Eq. (22) products of the $\overline{N}$'s. Consequently, Eq. (22) simplifies to the expression

$$1/\tau_q^{(2,2)e} = \sum_{kk'} (16\pi/\hbar)\,[(4/5)\,\eta^2 I N^{-1} k_F^{-2} \vec{k}\cdot\vec{k}']^2 \times$$

$$\times \overline{N}_{k+k'-q}\,\delta(E_q + E_{k+k'-q} - E_k - E_{k'}). \tag{34}$$

Elementary calculations give the simple result (*cf.* [5] for a similar result derived for the Heisenberg ferromagnet)

$$1/\tau_q^{(2,2)e} = \frac{36}{25}\pi^{3/2}\,\frac{n^2 I^2}{\hbar\alpha k_F^2}\,\eta^4 (q/k_F)^3 \left(\frac{k_B T}{\alpha k_F^2}\right)^{5/2} Z_{5/2}(\varepsilon) \tag{35}$$

where $\varepsilon = 2\mu_B H/k_B T$. The auxiliary function standing in Eq. (35) is defined as $Z_n(\varepsilon) = \sum_{k=1}^{\infty} k^{-n} \exp(-k\varepsilon)$. For $\varepsilon = 0$ it reduces to the Riemann ζ-function, $Z_n(0) = \zeta(n)$.

In concluding the present Section it should be noted that the general pattern of magnon relaxation phenomena is to a much extent model-independent. In particular, if in our itinerant electron model we calculate the coefficients (5) and (6) of the effective magnon

Hamiltonian in lowest order with respect to magnon wave vectors, we obtain exactly the same results for the dipolar part of the magnon inverse relaxation time as those known for the Heisenberg ferromagnet (compare Eqs (24) and (25) with results quoted in reference [5]). The lowest order electrostatic contribution to the inverse relaxation time is the same in both models, allowing only for the obvious difference in the interaction parameter. In particular, Eq. (35) corresponds to the low-temperature expression for the magnon relaxation time in localized-spin ferromagnets quoted in [5]. The formula (33) for low-energy magnons is similar to the result obtained for the Heisenberg model in [12] by an approach different from the kinetic equation method used in the present paper. The leading term of the right-hand side of Eq. (33) was calculated for the itinerant electron ferromagnet in [11].

4. Corrections to the spontaneous magnetization from magnon interaction

It was shown for the first time by Dyson [6] that for the Heisenberg ferromagnet the interaction of magnons contributes a correction to the expansion of the magnetization in powers of temperature which is proportional to the fourth power of temperature. A similar result follows also for itinerant-electron ferromagnets [7]. Here we shall rederive and slightly generalize this result starting from the effective Hamiltonian and adapting the scheme of calculations developed for the Heisenberg ferromagnet.

For calculating corrections to the spontaneous magnetization from magnon interaction we shall neglect here the dipolar interaction. This simplification means that the results will be inaccurate in the region of extremely low temperatures, say below 1°K. We start with the simplified effective Hamiltonian

$$\mathscr{H}_e = \sum_q E_q \beta_q^+ \beta_q + \sum_{kk'q} \Gamma_{kk'}^q \beta_{k+q}^+ \beta_{k'-q}^+ \beta_{k'} \beta_k. \tag{36}$$

For the Heisenberg ferromagnet, corrections to the spontaneous magnetization from magnon interaction were calculated by many authors (see *e. g.* [5] for a review); most of the calculations were based on the fundamental paper by Dyson [6]. Here we shall use quite formally the formulae quoted in [8]. They are based on a modification of the Dyson theory due to Szaniecki [9] and take for granted the fact proved by Dyson [6] that the kinematical interaction of spin waves has no influence on the low-temperature thermodynamic functions of the Heisenberg ferromagnet.

For the itinerant-electron ferromagnets a similar problem of kinematical interaction of magnons is much more difficult and in this respect the spin wave theories in the two models have the deepest differences. At present there are no results available concerning something like kinematical interaction of magnons in the itinerant electron ferromagnets. Any exact approach along a reasoning similar to that known for the Heisenberg model will be hindered by the fact that the operators β_q^+, β_q identified as magnon creation and annihilation operators have complicated commutation properties which only in the Random Phase Approximation reduce to the boson commutation rules. Further, the expression (36) for the effective Hamiltonian (36) is not exact, contrary to Dyson's theory in which the magnon Hamiltonian represents exactly the exchange energy, but is based on several approximations: it rests on the RPA, neglects terms of higher than the fourth order with respect to the magnon operators, to say nothing of the neglect of the magnon--electron interaction which underlies the method of the effective Hamiltonian.

However, there are good reasons to believe that in the region of low temperatures in which the number of thermally excited magnons is low the effect of all the above--mentioned limitations, which can be considered as a substitute for kinematical interaction of magnons in itinerant electron ferromagnetics, is negligible as far as thermodynamic properties are concerned. Therefore, from now on we shall use the results described in [8] starting from the effective Hamiltonian $\mathscr{H}_e$, (36) which has the same form as the Hamiltonian (6.6–8) of [8] (we shall quote the formula (6.6) of reference [8] as (SW-6.6), *etc.*). In order to use the formulae of [8] we have to put $\Gamma^q_{kk'}$ defined by Eq. (11) instead of the expression $(-\frac{1}{2} JN^{-1} \Gamma^q_{kk'})$ which appear in the interaction Hamiltonian (SW–6.8) and the following formulae of [8]. Let $\mathscr{Z} = \mathrm{Tr}\exp(-\beta\mathscr{H}_e)$ denote the partition function of the system of magnons and $\mathscr{Z}_0 = \mathrm{Tr}\exp(-\beta\sum_q E_q\beta^+_q\beta_q)$ the partition function for non-interacting magnons. Calculation of $\mathscr{Z}_0$ for any degree of accuracy is trivial.

Using the Matsubara [10] perturbation expansion the partition function can be written in the form (*cf.* SW–6.67; 6.70)

$$\mathscr{Z} = \mathscr{Z}_0 \exp \sum_{m=1}^{\infty} D_m. \tag{37}$$

The leading terms in the perturbation series $\sum_{m=1}^{\infty} D_m$ are proportional to the fourth power of temperature. According to the formulae (SW–6.72; 6.74, 6.76, 6.77), in lowest order with respect to temperature (*i. e.*, neglecting terms of higher order than T^4) we have

$$D_1 = -2\beta \sum_{kk'} \overline{\Gamma}^0_{kk'} \overline{N}_k \overline{N}_{k'}, \tag{38}$$

$$D_2 = 2^2\beta \sum_{kk'q} \overline{\Gamma}^q_{kk'} \overline{\Gamma}^{-q}_{k-q,k'+q} (E_{k'+q}+E_{k-q}-E_k-E_{k'})^{-1} \overline{N}_k \overline{N}_{k'} + O(\beta^{-5}), \tag{39}$$

and, in general,

$$D_n = (-2)^n\beta \sum_{\substack{kk' \\ q_1,q_2,\dots,q_{n-1}}} \overline{\Gamma}^{q_1}_{kk'} \overline{\Gamma}^{-q_1+q_2}_{k-q_1,k'+q_1} \overline{\Gamma}^{-q_2+q_3}_{k-q_2,k'+q_2} \dots \overline{\Gamma}^{-q_{n-1}}_{k-q_{n-1},k'+q_{n-1}} \times$$
$$\times (E_{k-q_1}+E_{k'+q_1}-E_k-E_{k'})^{-1}(E_{k-q_2}+E_{k'+q_2}-$$
$$-E_k-E_{k'})^{-1} \dots (E_{k-q_{n-1}}+E_{k'+q_{n-1}}-E_k-E_{k'})^{-1} \overline{N}_k \overline{N}_{k'} + O(\beta^{-5}). \tag{40}$$

Here, as in Section 3, $\overline{N}_k$ denotes the magnon occupation number at thermal equilibrium.

Calculation of D_1 is straightforward. For low temperatures only small values of wave vectors contribute to the sum in Eq. (38). The result of standard calculations is

$$D_1 = -\frac{3\pi}{5}\left(1+\frac{54}{35}\eta\right) Nn^2\eta^3(I/\alpha k_F^2) \times$$
$$\times [Z_{5/2}(2\mu_B H/k_B T)]^2 (k_B T/\alpha k_F^2)^4 + O(T^5). \tag{41}$$

In order to calculate D_2, D_3 *etc.* we have to sum over all values of the wave-vector q from the first Brillouin zone. Unfortunately, the magnon energy E_k and the interaction coefficient $\bar{\Gamma}^q_{kk'}$ are given only as expansions in powers of the wave-vectors components which are valid only in the central region of the first Brillouin zone. For the Heisenberg model the main contributions to the sums over q's in the right-hand sides of Eqs (38), (39), (40) come from the region near the centre of the first Brillouin zone. We expect the same to be true for the itinerant-electron model and, in computing D_m for $m \geqslant 2$ we adopt an approximation like Debye's for phonons. We shall restrict sums over q to values satisfying the condition $|\vec{q}| < q_0$, where the parameter q_0 is the maximal value of the wave vector for which the simple quadratic dispersion relation for magnons, $E_q = 2\mu_B H + \alpha q^2$, holds with reasonable accuracy. This definition is a little vague but the final result is not too much sensitive to the exact value of q_0. In practice we can take for q_0 a value between $\frac{1}{2}(\pi/a)$ (a is the lattice constant) and the Debye wave vector.

Adopting the above approximation we can calculate D_m, $m \geqslant 2$, in the standard way and the result is

$$D_m = \frac{9\pi}{8}[Z_{5/2}(2\mu_B H/k_B T)]^2 Nn(k_F/q_0)^5 \times$$
$$\times \{\tfrac{2}{5} n\eta(I/\alpha k_F^2)(q_0/k_F)^3\}^m (k_B T/\alpha k_F^2)^4 + O(T^5). \tag{42}$$

In calculating D_m we imposed the condition $\eta(q_0/k_F)^2 \ll 1$, in order to simplify calculations.

Finally, the correction to the free energy of the system of magnons due to the magnon interaction is

$$\Delta F = -\beta^{-1} \ln(\mathscr{Z}/\mathscr{Z}_0) = -\beta^{-1} \sum_{m=1}^{\infty} D_m. \tag{43}$$

As is evident from Eq. (42), $\sum_{m=1}^{\infty} D_m$ is a series which can be easily summed up giving in the lowest order with respect to temperature

$$\sum_{m=1}^{\infty} D_m = -\frac{3\pi}{5}[Z_{5/2}(2\mu_B H/k_B T)]^2 Nn^2\eta^3(I/\alpha k_F^2) \times$$
$$\times \left\{1 - \frac{6}{5}\eta\left[\frac{2}{7} + \frac{n(I/\alpha k_F^2)(q_0/k_F)}{1 - \frac{4}{5}\eta^2(I/\alpha k_F^2)(q_0/k_F)^3}\right\} \times$$
$$\times (k_B T/\alpha k_F^2)^4 + O(T^5). \tag{44}$$

The result (44) was obtained under the restriction $(4/5)\,\eta^2(I/\alpha k_F^2)(q_0/k_F)^3 < 1$. This condition, together with the former one, $\eta(q_0/k_F)^2 \ll 1$, limits the validity of the present approach.

From Eqs (43) and (44) it is easy to derive other thermodynamic quantities. For instance, the correction to the spontaneous magnetization due to magnon interaction is

$$M(T)-M_0(T) = -\frac{6\pi}{5}\zeta\left(\frac{5}{2}\right)\zeta\left(\frac{3}{2}\right)n\eta^3(I/\alpha k_F^2)\times$$

$$\times\left\{1-\frac{6}{5}\eta\left[\frac{2}{7}+\frac{n(I/\alpha k_F^2)q_0/k_F}{1-\frac{4}{5}\eta^2(I/\alpha k_F^2)(q_0/k_F)^3}\right]\right\}\times$$

$$\times(k_BT/\alpha k_F^2)^4\,M(0)+O(T^5). \qquad (45)$$

$M(T)$ denotes the spontaneous magnetization at temperature T, its value at $T = 0$ is the saturation magnetization $M(0) = 2\mu_B Nn$, and $M_0(T)$ is the spontaneous magnetization at temperature T as calculated from the linear spin wave theory. $M_0(T)$ is given by the usual expansion containing terms proportional to $T^{3/2}$, $T^{5/2}$, $T^{7/2}$ *etc.* (*cf.* [6]).

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