KIRCHHOFFIAN DIFFRACTION OF LONGITUDINAL ELASTIC WAVES

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The paper deals with the ambiguity of the tensor potential appearing in Kirchhoff's theory of elastic wave diffraction. Only the case of a longitudinal elastic wave emerging from an isotropic point source is considered. It is shown that a certain tensor $\hat{G}$ can be separated from the tensor potential $\hat{W}$ of Gniadek (*Acta Phys. Polon.*, **31**, 443 (1967)) for this wave. In the expression for the diffracted wave the edge integral of the tensor $\hat{G}$ vanishes, but its presence causes the elementary contribution $\hat{\boldsymbol{W}}\boldsymbol{ds}$ of the element $\boldsymbol{ds}$ of the diffracting edge to violate the equations of motion. After removing the tensor $\hat{G}$ from the tensor potential $\hat{W}$ the diffracted wave is presented as a superposition of two longitudinal and three transverse waves. Calculations of the wave were carried out by the stationary phase method for different points of observation. The differences between the diffracted wave due to the tensor potential $\hat{W}$ (Gniadek, *Acta Phys. Polon.*, **36**, 331 (1969)) and that obtained in this work with the use of the tensor potential $\hat{W}' = \hat{W} - \hat{G}$ are discussed.

1. Introduction

The application of Kirchhoff's theory of diffraction to elastic media has been treated in the papers by Petykiewicz (1966) and Gniadek (1967). The latter gives a form of tensor potential, essential when using Young's interpretation of diffraction phenomenon for the elastic waves.

A closer analysis of this tensor potential shows that in the case of an incident longitudinal wave it contains a certain tensor $\hat{G}$ in implicit form. Each row of this tensor is the gradient of the appropriate vector potential $\boldsymbol{W}_0$ of Rubinowicz taken over the coordinates of the running point Q on the diffracting edge B. The presence of the tensor $\hat{G}$ in implicit form complicates the explanation of the action of the elementary laws at the formation of a diffracted wave. It is known that in the scalar case the contribution coming from an element of arc of the diffracting edge does not generally have to represent an elementary law. The integrand appearing in the formula for the diffracted wave — hence, the elementary law also — is given with an accuracy of up to the gradient of the scalar function (*cf.* Rubinowicz 1969). This aspect of the theory has been analyzed in the elastic case.

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In particular, the problem of choosing the initial formula for calculating the wave by the stationary phase method was considered. As a result of the analysis performed, the tensor $\hat{G}$ mentioned above was removed from the potential given by Gniadek (1967) in the case when the longitudinal wave emerges from an isotropic point source (*cf.* also Gniadek 1969). By means of the tensor potential modified in this way the diffracted wave was obtained.

It proved that the integrand in the formula for the diffracted wave represents several kinds of elementary spherical waves. After this finding, the diffracted wave was established to be the sum of two longitudinal waves and three transverse waves, concordant in phase, but of different polarization.

When calculating the diffracted wave by the stationary phase method some computations of Gniadek (1969) were repeated. An analysis was made of the differences which occur between the diffracted wave given by the integral $\oint_B \hat{W} d\boldsymbol{s} = \boldsymbol{u}_B$ and that given by the integral $\boldsymbol{u}^{(D)} = \oint_B (\hat{W} - \hat{G}) d\boldsymbol{s}$ obtained in this work. It is found that in first approximation the calculations of the diffracted wave by the two formulae in mention give the same result. The difference between them is brought down to the terms containing the wave number in powers lower than the leading term. This is true for the whole range of observation. Both of these formulae show (in agreement with the required conservation of wave motion continuity) an identical jump of amplitude, equal in value to the amplitude of the incident wave at the shadow boundary. The mentioned differences only concern the longitudinal part in the diffracted wave. The results pertinent to the transverse parts are identical in the entire range of observation.

2. *Modified form of the tensor potential for the longitudinal wave and the diffracted wave relevant to it*

If in the Kirchhoff-Huygens principle we replace the scalar fields in a formal way by certain vector fields $\boldsymbol{u}(Q)$, and substitute the normal differentiation operator $\boldsymbol{n}$ by the stress operator

$$\tilde{T}_Q = 2\mu \frac{\partial}{\partial n} + \lambda \boldsymbol{n}\, \mathrm{div} + \mu(\boldsymbol{n} \times \mathrm{curl})$$

and the function $f = e^{ikr}/r$ by the tensor $\hat{\Gamma}$ of Kupradze (1950), we get the Huygens principle for elastic waves in the Kupradze formulation (1950, 1963):

$$\boldsymbol{u}(P) = \frac{1}{4\pi} \int_F [\hat{\Gamma}(P, Q)\tilde{T}_Q \boldsymbol{u}(Q) - \boldsymbol{u}(Q)\tilde{T}_Q \hat{\Gamma}(P, Q)] df.$$

The notation used is as follows. λ and μ are Lamé constants, $\boldsymbol{n}$ is the normal going out of the surface F enclosing the region R, and r is the distance of the point P (inside region R) from the point of integration Q on surface F. The function $\boldsymbol{u}(Q)$ defines the state of the field at point Q and is a regular solution of the equation of motion for an elastic medium,

$$\frac{1}{k_1^2}\nabla \operatorname{div} \boldsymbol{u} - \frac{1}{k_2^2} \operatorname{curl} \operatorname{curl} \boldsymbol{u} = -\boldsymbol{u} \tag{2.1}$$

$k_1 = \omega/a$ and $k_2 = \omega/b$ are the wave numbers of the longitudinal and transverse elastic waves, a and b denoting their respective velocities. The result of the operation $\hat{T}_Q \boldsymbol{u}(Q)$ presents the stress acting on an element of surface df having an outgoing normal $\boldsymbol{n}$. The tensor $\hat{\Gamma}$ (defined below) fulfills the same role as the function e^{ikr}/r does in the scalar case. As proved by Petykiewicz (1966), the integrand in Kupradze's formulation of the Helmholtz-Huygens principle can be expressed as the normal component of a certain tensor field $\hat{\Omega}(P, Q)$. This component, being a function of the point of integration Q on the surface F surrounding the observation point P, is sourceless and may be written as the curl of a tensor potential $\hat{W}$.

If the function $\boldsymbol{u}(Q)$ satisfies Sommerfeld's conditions at infinity and Kirchhoff's conditions at the screen, the elastic displacement at point P is given by

$$\boldsymbol{u}(P) = \int_F \hat{Q}(P, Q)\boldsymbol{n}df_Q = \int_f \boldsymbol{n}\operatorname{Curl}_Q \hat{W}(P, Q)df_Q,$$

where f is the area of the aperture in the screen. Curl_Q stands for the curl of the tensor $\hat{W}$ (differentiation concerns the point of integration Q).

Making use of Stokes' theorem for the integral over f yields (Petykiewicz 1966)

$$\boldsymbol{u}(P) = \frac{1}{4\pi}\oint_B \hat{W}(P, Q)d\boldsymbol{s} + \frac{1}{4\pi}\sum_{j=1}^{n}\oint_{s_j} \hat{W}(P, Q)d\boldsymbol{s}_j.$$

The line integral over the contour B of the aperture in the screen is the diffracted wave and in the case of a spherical or plane incident wave represents Young's explanation of the diffraction phenomenon. The sum of the integrals over the singular regions of the potential $\hat{W}$ on the aperture surface is interpreted as being a geometrical wave (Miyamoto and Wolf 1962, Rubinowicz 1962).

We consider wave motion in an infinite, loss-less elastic medium. For an isotropic source of longitudinal waves placed at point L the elastic displacement $\boldsymbol{u}(Q, L)$ at any point Q of surface F is written in the form

$$\boldsymbol{u} = \nabla_Q f(\varrho) \tag{2.2}$$

where $f(\varrho) = e^{ik_1\varrho}/\varrho$, ϱ being the distance of the source L from the edge point Q of the contour B of the aperture in the screen.

For this form of incident wave the components of the tensor potential $\hat{W}(P, Q)$ (Gniadek 1967) are expressed as

$$W_{ij} = \frac{1}{4\pi}\Big\{2b^2(\boldsymbol{\Gamma}^{(i)} \times \boldsymbol{u}) + f_1(\boldsymbol{\delta}^{(i)} \times \boldsymbol{u}) - f(\varrho)(\boldsymbol{\delta}^{(i)} \times \nabla_Q f_1) + -4\pi \frac{\partial}{\partial x_i^i}(\boldsymbol{W}_0) + f(\varrho)(\boldsymbol{\delta}^{(i)} \times \nabla_Q f_2)\Big\}_j. \tag{2.3}$$

The tensor $\hat{\Gamma}$ consists of the sum of two tensors, $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$, the rows of which are presentable by the following vector fields (Kupradze 1950)

$$\Gamma_1^{(i)} = -\frac{1}{\omega^2}\nabla \operatorname{div} \delta^{(i)} f_1 \tag{2.3a}$$

$$\Gamma_2^{(i)} = \frac{1}{\omega^2} \operatorname{curl} \operatorname{curl} \delta^{(i)} f_2 \tag{2.3b}$$

$\delta^{(i)}$, with $i = 1, 2, 3$, denotes the respective versors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of the coordinate axes, and the superscript (i) at the same time denotes the i-th row of the given tensor. The functions $f_1 = e^{ik_1 r}/r$ and $f_2 = e^{ik_2 r}/r$, with r standing for the distance between the point of obser vation and the running point of the edge, are the spherical-symmetric solution of th vibration equation.

$$\boldsymbol{W}_0 = -\frac{1}{4\pi}\frac{e^{ik_1(r+\varrho)}}{r\varrho}\frac{\boldsymbol{r}\times\boldsymbol{\varrho}}{r\varrho+\boldsymbol{r\varrho}} \tag{2.3c}$$

is the known vector potential in the scalar case (Rubinowicz 1966). The capital letters L, P and Q at the differentiation operators denote differentiation over the coordinates of the given points: L — wave source, Q — point of integration on contour B of the aperture f in the screen, and P — point of observation.

In the scalar case the vector potential appearing in Young's interpretation of Kirchhoff's theory of diffraction is defined with an accuracy of up to the gradient of the scalar function. In the vector case the counterpart of such a gradient is a tensor which has gradients of certain scalar functions as rows. The presence of such a tensor in the form

$$\hat{G} = \begin{Vmatrix} (\nabla_Q \boldsymbol{i}\cdot \boldsymbol{W}_0)_x, & (\nabla_Q \boldsymbol{i}\cdot \boldsymbol{W}_0)_y, & (\nabla_Q \boldsymbol{i}\cdot \boldsymbol{W}_0)_z \\ (\nabla_Q \boldsymbol{j}\cdot \boldsymbol{W}_0)_x, & (\nabla_Q \boldsymbol{j}\cdot \boldsymbol{W}_0)_y, & (\nabla_Q \boldsymbol{j}\cdot \boldsymbol{W}_0)_z \\ (\nabla_Q \boldsymbol{k}\cdot \boldsymbol{W}_0)_x, & (\nabla_Q \boldsymbol{k}\cdot \boldsymbol{W}_0)_y, & (\nabla_Q \boldsymbol{k}\cdot \boldsymbol{W}_0)_z \end{Vmatrix} \tag{2.4}$$

is ascertained to be in the tensor potential (2.3). The quantities $\boldsymbol{i}\cdot\boldsymbol{W}_0 = W_{0x}$, $\boldsymbol{j}\cdot\boldsymbol{W}_0 = W_{0y}$ and $\boldsymbol{k}\cdot\boldsymbol{W}_0 = W_{0z}$ are the respective components of the vector potential (2.3c). In order to draw the tensor (2.4) out of the potential (2.3) we make use of the following conversions:

$$\frac{1}{4\pi}\{\delta^{(i)}\times[f_1\nabla_Q f(\varrho) - f(\varrho)\nabla_Q f_1]\} = \delta^{(i)}\times \operatorname{curl}_Q \boldsymbol{W}_0$$

and

$$\frac{\partial}{\partial x_L^i}(\boldsymbol{W}_0) = (\delta^{(i)}\cdot\nabla_L)\boldsymbol{W}_0 = \nabla_L(\delta^{(i)}\cdot\boldsymbol{W}_0) - \delta^{(i)}\times\operatorname{curl}_L \boldsymbol{W}_0.$$

In the grad and curl operations we shall replace differentiation over the source coordinates L by differentiation with respect to the edge point Q and point of observation P. In doig this we make use of the known formula for the derivative of a composite function of two variables, remembering that $\varrho = \boldsymbol{LQ}$ is independent of the coordinates x_P, y_P and z_P of the observation point P, while $\boldsymbol{r} = \boldsymbol{PQ}$ is independent of the coordinates x_L, y_L and z_L of the source L. Use is now made of the conversion expressions

$$\nabla_Q(\boldsymbol{\delta}^{(i)} \cdot \boldsymbol{W}_0) = -\nabla_P(\boldsymbol{\delta}^{(i)} \cdot \boldsymbol{W}_0) - \nabla_L(\boldsymbol{\delta}^{(i)} \cdot \boldsymbol{W}_0)$$

and

$$\boldsymbol{\delta}^{(i)} \times \operatorname{curl}_Q \boldsymbol{W}_0 = -\boldsymbol{\delta}^{(i)} \times [\operatorname{curl}_P \boldsymbol{W}_0 + \operatorname{curl}_L \boldsymbol{W}_0]$$

for getting the identity

$$-\frac{1}{4\pi}[f_1 \nabla_Q f(\varrho) - f(\varrho)\nabla_Q f_1] \times \boldsymbol{\delta}^{(i)} - \frac{\partial}{\partial x_L^i}\boldsymbol{W}_0 = \boldsymbol{\delta}^{(i)} \times [\operatorname{curl}_Q \boldsymbol{W}_0 +$$

$$+ \operatorname{curl}_L \boldsymbol{W}_0] - \nabla_L(\boldsymbol{\delta}^{(i)} \cdot \boldsymbol{W}_0) = \nabla_Q(\boldsymbol{\delta}^{(i)} \cdot \boldsymbol{W}_0) + (\boldsymbol{\delta}^{(i)} \cdot \nabla_P)\boldsymbol{W}_0. \quad (2.5)$$

With the identity (2.5) we shall separate from (2.3) the explicit form of the tensor $\hat{G}$ (2.4). The components of the tensor potential $\hat{W}'$ can now be written in the form

$$W'_{ij} = \frac{1}{4\pi}\{2b^2(\boldsymbol{\Gamma}^{(i)} \times \boldsymbol{u}) + 4\pi(\boldsymbol{\delta}^{(i)} \cdot \nabla_P)\boldsymbol{W}_0 + f(\varrho)\boldsymbol{\delta}^{(i)} \times \nabla_Q f_2 + 4\pi\nabla_Q(\boldsymbol{\delta}^{(i)} \cdot \boldsymbol{W}_0)\}_j \quad (2.5a)$$

where $\nabla_Q(\boldsymbol{\delta}^{(i)} \cdot \boldsymbol{W}_0)|_j = G_{ij}$ is the appropriate component of the tensor (2.4). We get the diffracted wave by integrating (2.5a) over the contour B of the aperture in the screen. Owing to the known theorem of integral calculus, $\oint \hat{G}\boldsymbol{ds}$ vanishes, and the removal of the tensor $\hat{G}$ from the tensor potential (2.3) or (2.5a) bears no effect on the result of the integration.

In the description of the wave motion as a whole the forms of the potential given by (2.3) or (2.5a) and the tensor potential

$$\hat{W}' = \hat{W} - \hat{G} \quad (2.6)$$

are quite equivalent. But when formulating the elementary laws and when applying the stationary phase method, the potential (2.6) is the correct one. In order to justify this closer, we shall analyze this problem along lines similar to those used in the scalar case.

In general, instead of a tensor potential $\hat{W}'(P, Q)$ we may make use of the potential

$$\hat{W}(P, Q) = \hat{W}'(P, Q) + \operatorname{grad}_Q \boldsymbol{F}(P, Q)$$

for obtaining the diffracted wave, where the vector function $\boldsymbol{F}(P, Q)$ is an arbitrary function in the general case.

However, if we want to comply strictly with Young's point of view we must require the contributions from the gradient to fulfill some proper differential equations. Otherwise, the contribution $\hat{W}(P, Q)\,\boldsymbol{ds}_Q$ coming from the edge element ds_Q cannot be interpreted as being wave motion. Apart from this, the function $\boldsymbol{F}(P, Q)$ is indeed quite arbitrary. In the case when we are calculating the diffracted wave by integrating over a closed contour, the shape of the function $\boldsymbol{F}(P, Q)$ is immaterial. In practice, we do not observe the various wavelets, but only the aggregate outcome of their superposition. It is only important for the whole to satisfy the equation of motion, which undoubtedly is true in the case of diffracted waves obtained with (2.3) and (2.6) (see below, also Appendix 3).

It is a different matter when the diffracted wave is calculated by the stationary phase method. In this case we integrate around the neighbourhood of the active edge point Q and the various contributions from the gradients do not have to vanish.

Application of the stationary phase method emphasizes Young's approach to the problem. Before employing it, however, it will be convenient to separate the various elementary waves out. To this aim, we shall express the integrand in the diffracted wave as a sum of irrotational and sourceless fields (Eq. (2.8) *et seq.*). Of course, when a contribution from a gradient cannot be presented in the form of at least one of such fields, and in addition it does not fulfill the equations of motion, it cannot be considered as a wave — particularly a Young wave.

Rubinowicz demonstrated in the scalar case (1969) that the contributions from the gradient are not observed in practice. On the other hand, the purposefulness of calculating the integral by asymptotic methods may be questioned, as its accurate value is known; in our explicit case it is equal zero. From (2.6) we have

$$\oint_B \hat{W} ds = \oint_B \hat{W}' ds + \oint_B \hat{G} ds.$$

It is shown in Appendix 4 that the elementary contribution $\hat{G}ds$ does not fulfill the equation of motion (2.1), but the contribution $\hat{W}'ds$ does, as follows from the form of the diffracted wave given below (see also Appendix 3). Consequently, $\hat{W}ds$ does not fulfill the equation of motion and cannot be presented as an elementary spherical wave emerging from the element ds, as is required by Young in his approach. Hence, for interpretational reasons and because it is necessary to consider the case when the diffracted wave comes from an arc of the regular curve $\widehat{AB}(A \neq B)$, we shall use the tensor potential (2.6) for finding the diffracted wave. In doing this, we take advantage of the identities proved in Appendices 1 and 2:

$$(\hat{\Gamma}_1 \times \boldsymbol{u})\boldsymbol{s}_0 = \frac{1}{\omega^2}\frac{e^{ik_1(r+\varrho)}}{r\varrho}[A_1(\boldsymbol{s}_0 \times \boldsymbol{\varrho}_0) - B_1[(\boldsymbol{s}_0 \times \boldsymbol{\varrho}_0)\cdot \boldsymbol{r}_0]\boldsymbol{r}_0] =$$

$$= -\frac{1}{\omega^2}\nabla_P[(\nabla_P f_1 \times \boldsymbol{u})\cdot \boldsymbol{s}_0] \tag{2.7}$$

$$(\hat{\Gamma}_2 \times \boldsymbol{u})\boldsymbol{s}_0 = -\frac{1}{\omega^2}\operatorname{curl}_P[(\boldsymbol{u} \times \boldsymbol{s}_0) \times \nabla_P f_2] =$$

$$= \frac{1}{\omega^2}\frac{e^{i(k_1\varrho + k_2 r)}}{r\varrho}\{A_2(\boldsymbol{s}_0 \times \boldsymbol{\varrho}_0) + B_2 \boldsymbol{r}_0 \times [\boldsymbol{r}_0 \times (\boldsymbol{s}_0 \times \boldsymbol{\varrho}_0)]\} \tag{2.7a}$$

$\boldsymbol{s}_0$ is the versor tangent to the element ds of the edge, and $\boldsymbol{r}_0$ and $\boldsymbol{\varrho}_0$ are the versors of the vectors $\boldsymbol{r}$ and $\boldsymbol{\varrho}$, respectively. A_i and B_i denote the following functions:

$$A_1 = \frac{1}{r}\left(ik_1 - \frac{1}{r}\right)\left(ik_1 - \frac{1}{\varrho}\right) \tag{2.7b}$$

$$B_1 = \left[\frac{3}{r}\left(ik_1 - \frac{1}{r}\right) + k_1^2\right]\left(ik_1 - \frac{1}{\varrho}\right) \tag{2.7b'}$$

$$A_2 = \frac{2}{r}\left(ik_2 - \frac{1}{r}\right)\left(ik_1 - \frac{1}{\varrho}\right) \tag{2.7c}$$

$$B_2 = \left[\frac{3}{r}\left(ik_2 - \frac{1}{r}\right) + k_2^2\right]\left(ik_1 - \frac{1}{\varrho}\right). \tag{2.7c'}$$

With the aid of these identities we put the diffracted wave in the form:

$$u^{(D)} = \nabla_P \oint_B (s_0 \cdot W_0) ds - \frac{1}{2\pi k_2^2} \nabla_P \oint_B [s_0 \cdot (\nabla_P f_1 \times u)] ds -$$

$$- \frac{1}{4\pi} \operatorname{curl}_P \oint_B s_0 f_2 f(\varrho) ds + \frac{1}{2\pi k_2^2} \operatorname{curl}_P \oint_B [(u \times s_0) \times \nabla_P f_2] ds. \tag{2.8}$$

It is shown in Appendix 3 that each of the integrals in (2.8) individually fulfills the equation of motion. Hence, as had already been mentioned, formula (2.8) represents the following set of diffracted waves.

The first two integrals give the longitudinal waves,

$$u_{\text{long}}^{(D)} = \oint_B \nabla_P(s_0 \cdot W_0) ds - \frac{1}{2\pi k_2^2} \oint_B \frac{e^{ik_1(r+\varrho)}}{r\varrho} \{A_1(s_0 \times \varrho_0) - B_1[(s_0 \times \varrho_0) \cdot r_0] r_0] ds -$$

$$- B_1[(s_0 \times \varrho_0) \cdot r_0] r_0\} ds \tag{2.9}$$

determined by the phase factor $e^{ik_1(r+\varrho)}$. The elementary wave in (2.9) is the superposition of a wave, which is continuous in the entire region of observation, and a wave containing the potential W_0. The presence of the function W_0 indicates there is a jump of the wave (2.9) at the shadow boundary. The shadow boundary is defined as the lateral surface of the truncated cone generated by the rays LQ belonging to the pencil of straight lines emerging from L and passing through the diffracting edge.

The third integral in (2.8) represents the simple case of a transverse wave,

$$u_{tr_1}^{(D)} = -\frac{1}{4\pi} \oint_B \frac{e^{i(k_2 r + k_1 \varrho)}}{r\varrho} \frac{1}{r}\left(ik_2 - \frac{1}{r}\right)(r_0 \times s_0) ds. \tag{2.9a}$$

The elementary displacement in (2.9a) at point P is always perpendicular to the edge element ds, wherein the maximum amplitude is at the points of observation at which $r \perp s_0$. This kind of wave is independent of the direction of incidence of the wave from the source L.

We can write the last of the integrals as a sum of two integrals, each of which fulfills the equation of motion. We present it as the superposition of two transverse waves:

$$u_{tr_2}^{(D)} = \frac{1}{2\pi k_2^2} \oint_B \frac{e^{i(k_2 r + k_1 \varrho)}}{r\varrho} [A_2(s_0 \times \varrho_0) + B_2 r_0 \times [r_0 \times (s_0 \times \varrho_0)]] ds. \tag{2.9b}$$

Let us notice here that the appearance of three transverse waves and two longitudinal waves in the diffraction process is evidence that the diffraction process is more complex than the processes taking place in reflection, also as regards the number of constituents which appear.

3. *Calculation of the diffracted wave by the stationary phase method*

We shall apply the stationary phase method to the diffracted wave presented in the form resulting from a combination of formulae (2.9), (2.9a) and (2.9b), *viz.*,

$$u^{(D)} = -\frac{1}{4\pi} \oint \nabla_P \frac{e^{ik_1(r+\varrho)}}{r\varrho} \frac{(r \times \varrho) \cdot s_0}{r\varrho + r\varrho} ds - \frac{1}{2\pi k_2^2} \oint \frac{e^{ik_1(r+\varrho)}}{r\varrho} \{A_1(s_0 \times \varrho_0) -$$

$$- B_1[(s_0 \times \varrho_0) \cdot r_0]\} ds - \frac{1}{4\pi} \oint \frac{e^{i(k_1\varrho + k_2 r)}}{r\varrho} \left(ik_2 - \frac{1}{r}\right)(r_0 \times s_0) ds +$$

$$+ \frac{1}{2\pi k_2^2} \oint \frac{e^{i(k_2 r + k_1 \varrho)}}{r\varrho} \{A_2(s_0 \times \varrho_0) + B_2 r_0 \times [(\varrho_0 \times s_0) \times r_0]\} ds. \tag{3.1}$$

The presence of the potential W_0 requires the region of observation in which calculations are to be performed to be established prior to using the stationary phase method. This is necessary because the singularities which appear when the direction of observation is a prolongation of the direction of wave incidence on the element ds have to be removed from the potential W_0. This allows the region of observation near the shadow boundary to be defined as the set of points P at which r_0 can be replaced by $-\varrho_0$ without making a big error. In the regions of observation where this cannot be done owing to excessive error, the influence of the singularities (the Fresnel integrals) on the value of the amplitude is inappreciable. In this case the stationary phase method may be applied directly to formula (3.1). These differences in procedure only concern the first integral in (2.8) or (3.1), as the remaining terms are continuous on the shadow boundary. If the observation point is placed far from the shadow boundary, formula (3.1) may be substituted by the asymptotic approximation used for large wave numbers. We disregard the terms containing the coefficients A_1, Eq. (2.7b), and A_2, Eq. (2.7c), because they are smaller by the order of k than the other ones.

Let us also note that in first approximation

$$\oint_B \nabla_P(s_0 \cdot W_0) ds \approx -ik_1 \oint_B r_0 du^{(D)} \tag{3.2}$$

where

$$du^{(D)} = -\frac{e^{ik_1(r+\varrho)}}{4\pi r\varrho}\,\frac{(\boldsymbol{r}\times\boldsymbol{\varrho})\cdot \boldsymbol{s}_0}{r\varrho+\boldsymbol{r\varrho}}\,ds \tag{3.2a}$$

is the elementary diffracted wave in the scalar case. Moreover, from (2.7b′) and (2.7c′) we have

$$B_1 \approx ik_1^3\,\frac{e^{ik_1(r+\varrho)}}{r\varrho} \tag{3.3a}$$

and

$$B_2 \approx ik_1k_2^2\,\frac{e^{i(k_1\varrho+k_2r)}}{r\varrho}. \tag{3.3b}$$

Bearing all this in mind, we finally get from formula (3.1) the following form for the diffracted wave:

$$\boldsymbol{u}^{(D)} \approx -ik_1\oint_B \boldsymbol{r}_0 du^{(D)} + \frac{ik_1}{2\pi\kappa^2}\oint \frac{e^{ik_1\zeta}}{r\varrho}[(\boldsymbol{r}_0\times\boldsymbol{\varrho}_0)\cdot\boldsymbol{s}_0]\boldsymbol{r}_0 -$$

$$-\frac{ik_2}{4\pi}\oint_B \frac{e^{ik_2\vartheta}}{r\varrho}(\boldsymbol{s}_0\times\boldsymbol{r}_0)ds + \frac{ik_1}{2\pi\kappa}\oint_B \frac{e^{ik_2\vartheta}}{r\varrho}\{\boldsymbol{r}_0\times[\boldsymbol{r}_0\times(\boldsymbol{s}_0\times\boldsymbol{\varrho}_0)]\}ds \tag{3.4}$$

in which

$$\zeta = r+\varrho; \qquad \vartheta = \frac{1}{\kappa}\varrho + r$$

and $\kappa = \dfrac{k_2}{k_1}$ denotes the ratio of longitudinal wave velocity to the transverse wave velocity.

Formula (3.4) presents the approximate field of the diffracted wave, generated by the wave in the form (2.2). Owing to the way in which it was derived, we shall temporarily limit further considerations to observation points placed far away from the shadow boundary. As is shown later, once the stationary phase method is applied to (3.4) these limitations are found to be quite unnecessary.

Formula (3.4) is simple and easy to interpret, and we shall use it: 1° to discuss further the influence of the ambiguity of the tensor potential (2.3) (see Sec. 1) on the shape of the asymptotic expression for the sought diffraction field, and 2° to simplify calculations of the diffracted wave at observation points distant from the shadow boundary.

All this calls for a closer analysis of the approximation of (3.2) leading to formula (3.4). It must be emphasized that we are restricting the use of these approximate formulae to such diffraction fields for which the stationary phase method may be applied and, consequently, the asymptotic approximation for large wave numbers.

After executing the operations $\nabla_p\hat{\boldsymbol{W}}\boldsymbol{ds}$ we get terms having singularities in the form of the factor

$$\frac{1}{r\varrho+\boldsymbol{r\varrho}} \quad \text{or} \quad \frac{1}{(r\varrho+\boldsymbol{r\varrho})^2}.$$

These singularities are encountered wherever the denominator $(r\varrho+\boldsymbol{r\varrho})$ tends to zero. In our case this is near the shadow boundary. Their effect on the value of the field in this region might be considerable even though they could be disregarded owing to the order of the wave number which appears. The approximation of (3.2) consists in rejecting all terms which contain the singularities $\frac{1}{(r\varrho+\boldsymbol{r\varrho})^2}$. None the less, after applying to these terms an appropriate procedure (Petykiewicz 1963) and then the stationary phase method it can be shown that their influence on the diffraction field tends to zero when the point of observations approaches the shadow boundary (see Eq. (3.8) below). This lets us compare in Subsection a) of this section the results of calculations of (3.4) near the shadow boundary with the diffracted wave calculated from formula (3.1) for the same region of observation.

Let us now consider points of observation placed far away from the shadow boundary. In this region the effect of singularities associated with the term $\nabla_P \hat{W} \boldsymbol{s}_0$ does not become manifest and it does not matter whether we apply the stationary phase method to (3.1) and then make the approximation for large wave numbers or, as we do here, apply the stationary phase method directly on formula (3.4).

For the sake of comparison with (3.4) and subsequent formulae, we give here the formula of Gniadek (1969) for the diffracted wave when the incident wave is described by (2.2a), *viz.*,

$$\boldsymbol{u}_{(B)} = \frac{1}{4\pi}\oint\limits_B \{2b^2(\hat{\Gamma}_1 \times \nabla_P f(\varrho))\boldsymbol{s}_0 + f(\varrho)(\boldsymbol{s}_0 \times \nabla_Q f_1) - f_1(\boldsymbol{s}_0 \times \nabla_Q f(\varrho) -$$

$$-4\pi\nabla_L(\boldsymbol{W}_0 \cdot \boldsymbol{s}_0)\}ds + \frac{1}{4\pi}\oint\limits_B [2b^2(\hat{\Gamma}_2 \times \nabla_Q f(\varrho))\boldsymbol{s}_0 - f(\varrho)(\boldsymbol{s}_0 \times \nabla_Q f_2)]ds. \tag{3.6}$$

In order to approximate (3.6) for the case of large wave numbers we make use of Eqs (3.3a) and (3.3b) together with the following approximation

$$4\pi\nabla_L(\boldsymbol{W}_0 \cdot \boldsymbol{s}_0) \approx ik_1 \frac{e^{ik_1(r+\varrho)}}{r\varrho} \frac{(\boldsymbol{r}\times\boldsymbol{\varrho})\cdot\boldsymbol{s}_0}{r\varrho+\boldsymbol{r\varrho}} \boldsymbol{\varrho}_0$$

and get the counterpart of formula (3.4),

$$\boldsymbol{u}_{(B)} \approx ik_1 \oint\limits_B \boldsymbol{\varrho}_0 du^{(D)} + \frac{ik_1}{2\pi\kappa^2}\oint\limits_B \frac{e^{ik_1\zeta}}{r\varrho}[(\boldsymbol{r}_0\times\boldsymbol{\varrho}_0):\boldsymbol{s}_0]\boldsymbol{r}_0 ds + \frac{ik_1}{4\pi}\oint\limits_B \frac{e^{ik_1\zeta}}{r\varrho}(\boldsymbol{\varrho}_0 -$$

$$-\boldsymbol{r}_0)\times\boldsymbol{s}_0 ds - \frac{ik_2}{4\pi}\oint\limits_B \frac{e^{ik_2\vartheta}}{r\varrho}(\boldsymbol{s}_0\times\boldsymbol{r}_0)ds +$$

$$+\frac{ik_2}{2\pi\kappa}\oint\limits_B \frac{e^{ik_2\vartheta}}{r\varrho}\{\boldsymbol{r}_0\times[\boldsymbol{r}_0\times(\boldsymbol{s}_0\times\boldsymbol{\varrho}_0)]\}ds. \tag{3.6a}$$

To find the difference between the formula for the diffracted wave coming from the tensor potential in the form (2.3) and that in the form (2.6), we have to subtract (3.4) from (3.6a), which yields

$$\boldsymbol{u}^{(D)} - \boldsymbol{u}_{(B)} \approx -ik_1 \oint_B (\boldsymbol{r}_0 + \boldsymbol{\varrho}_0) du^{(D)} - \frac{ik_1}{4\pi} \oint_B \frac{e^{ik_1\zeta}}{r\varrho} (\boldsymbol{\varrho}_0 - \boldsymbol{r}_0) \times \boldsymbol{s}_0 ds. \tag{3.6b}$$

We shall show that when (3.6a) and (3.4) are computed by the stationary phase method this difference is reduced to zero. For this, we resort to the known representation of the vector field,

$$\boldsymbol{V}(P, Q) = \frac{1}{4\pi} (f_1 \nabla_Q f(\varrho) - f(\varrho) \nabla_Q f_1) =$$

$$= \frac{e^{ik_1(r+\varrho)}}{4\pi r\varrho} \left[\left(ik_1 - \frac{1}{\varrho} \right) \boldsymbol{\varrho}_0 - \left(ik_1 - \frac{1}{r} \right) \boldsymbol{r}_0 \right]$$

as the curl of the vector potential $\hat{\boldsymbol{W}}_0(P, Q)$ (Rubinowicz 1966):

$$\boldsymbol{V}(P, Q) = \mathrm{curl}_Q \, \hat{\boldsymbol{W}}_0$$

as is the case in formula (2.3c). It is easily seen that the integrand of the second integral in (3.6b) is the first approximation of the vector field

$$\boldsymbol{V}(P, Q) \approx ik_1 \frac{e^{ik_1(r+\varrho)}}{4\pi r\varrho} (\boldsymbol{\varrho}_0 - \boldsymbol{r}_0).$$

We substitute it by the first approximation of the function $\mathrm{curl}_Q \boldsymbol{W}_0$,

$$\mathrm{curl}_Q \, \boldsymbol{W}_0 \approx -ik_1 g[(\boldsymbol{r} \times \boldsymbol{\varrho}) \times (\boldsymbol{r}_0 + \boldsymbol{\varrho}_0)]$$

where

$$g = -\frac{1}{4\pi} \frac{e^{ik_1(r+\varrho)}}{r\varrho} \frac{1}{r\varrho + \boldsymbol{r}\boldsymbol{\varrho}}.$$

The difference (3.6b) then becomes

$$\boldsymbol{u}^{(D)} - \boldsymbol{u}_{(B)} \approx -ik_1 \oint_B g(\boldsymbol{r} \times \boldsymbol{\varrho}) \left[(\boldsymbol{r}_0 + \boldsymbol{\varrho}_0) \cdot \boldsymbol{s}_0 \right] ds. \tag{3.6c}$$

When we apply the stationary phase method we take account only of the contributions coming from the points Q_j of curve B at which the function $\zeta = r + \varrho$ takes on an extreme value, *i. e.*

$$\frac{dr}{ds} + \frac{d\varrho}{ds} = 0,$$

and this causes the integrand in (3.6c) to vanish.

Let us now notice that (3.6c) also represents the first approximation of $\oint\limits_B \hat{G} ds$ (*viz.* Appendix 4).

This means that applying the stationary phase method to (3.1) and (3.6) we obtain two asymptotical expansions whose first terms are identical. The difference appears only in the subsequent terms having $k_1^{-\frac{1}{2}}$.

We shall now look at the behaviour of the wave primarily in two regions of observation, namely, near and far from the shadow boundary.

a) The diffracted wave near the shadow boundary

A method for removing singularities in the vicinity of the shadow boundary in the case of vector fields was given by Petykiewicz (1965). We proceed quite analogously in the case of the integral

$$\oint\limits_B \nabla_P(\boldsymbol{s}_0 \cdot \boldsymbol{W}_0) ds.$$

Accomplishing the "grad$_P$" operation we have

$$\nabla_P(\boldsymbol{s}_0 \cdot \boldsymbol{W}_0) = g(\boldsymbol{s}_0 \times \boldsymbol{\varrho}_0) - \boldsymbol{s}_0(\boldsymbol{r} \times \boldsymbol{\varrho}_0)\left[g\left(ik_1 - \frac{1}{r}\right)\boldsymbol{r}_0 - \right.$$

$$\left. - g\,\frac{\boldsymbol{r}_0(r+\varrho)+\boldsymbol{R}}{r\varrho + \boldsymbol{r}\cdot\boldsymbol{\varrho}_0}\right] \tag{3.7}$$

where $\boldsymbol{R} = \boldsymbol{\varrho} - \boldsymbol{r} = \boldsymbol{LP}$. Since we are in the observation region where it may be assumed that $\boldsymbol{r}_0 = -\boldsymbol{\varrho}_0 = -\boldsymbol{R}_0$, hence,

$$\oint\limits_B \nabla_P(\boldsymbol{s}_0 \cdot \boldsymbol{W}_0) ds = -\oint\left\{g(\boldsymbol{s}_0 \times \boldsymbol{r}_0) + \boldsymbol{s}_0(\boldsymbol{r} \times \boldsymbol{\varrho})\left[\frac{1}{r}\, g\boldsymbol{\varrho}_0 - ikg\boldsymbol{\varrho}_0 + \right.\right.$$

$$\left.\left. + g\,\frac{\boldsymbol{\varrho}_0(r+\varrho)-\boldsymbol{R}}{r\varrho + \boldsymbol{r}\cdot\boldsymbol{\varrho}}\right]\right\} ds. \tag{3.7a}$$

For the integral in (3.7a) we use the same procedure as the mentioned author did, and get the following integral calculated by the stationary phase method:

$$4\pi \oint\limits_B \nabla_P(\boldsymbol{s}_0 \cdot \boldsymbol{W}_0) ds \approx -\sum_j \frac{\sqrt{(r+\varrho)-R}}{2r\varrho(r\varrho + \boldsymbol{r\varrho})}\,\frac{e^{i\left(k_1 R \pm \frac{\pi}{4}\right)}}{\sqrt{\zeta_j''}}\left\{(\boldsymbol{r}\times\boldsymbol{\varrho})\cdot\boldsymbol{s}_0\left[k_1\boldsymbol{\varrho}_0 + \right.\right.$$

$$\left.\left. + i\,\frac{\boldsymbol{\varrho}_0 + \boldsymbol{R}_0}{(r+\varrho+R)}\right] F(v) - [(\boldsymbol{r}\times\boldsymbol{\varrho})\cdot\boldsymbol{s}_0]\left[(r+\varrho-R)\frac{1}{r}\,\boldsymbol{\varrho}_0 + \boldsymbol{\varrho}_0 - \boldsymbol{R}_0\right]\right\}. \tag{3.8}$$

Here,

$$\xi = r+\varrho-R$$

$$F(v) = \int_{+\infty}^{\sqrt{\frac{k_1}{2\pi}(r+\varrho-R)}} e^{i\frac{\pi}{2}v^2} dv.$$

To the other terms of (3.1) we shall directly apply the stationary phase method (see Rubinowicz 1966).

In the case of a regular term describing a longitudinal wave, the integral appearing in the expression giving the contribution from a single critical point Q_j has the form

$$\int_{s-\Delta s_j}^{s+\Delta s_j} e^{ik_1[\zeta_j+\frac{1}{2}\zeta_j''(s-s_j)^2]} ds \approx \sqrt{\frac{2\pi}{k_1|\zeta_j''|}}\, e^{i\left(k_1\zeta_j\pm\frac{\pi}{4}\right)}$$

where $\Delta s_j = s-s_j$, with s_j being the coordinate of the point Q_j at which the phase function $\zeta = r+\varrho$ reaches maximum or minimum value.

In the case of the transverse waves (2.9a) and (2.9b) we likewise have for the point Q_l the integral

$$\int_{s-\Delta s_j}^{s+\Delta s_j} e^{ik_2[\vartheta_l+\frac{1}{2}\vartheta_l''(s-s_j)^2]} ds \approx \sqrt{\frac{2\pi}{k_2|\vartheta_l''|}}\, e^{i\left(k_2\vartheta_l\pm\frac{\pi}{4}\right)}$$

where s_l is the coordinate of the point Q_l at which the phase function acquires an extreme value.

Before proceeding to the formula for the diffracted wave near the shadow boundary we must make the following comments.

In the case of the longitudinal part of (2.9) of the diffracted wave, we shall restrict ourselves to the contribution from the critical point Q_j for which hold the relations:

$$(\boldsymbol{r}_j\times\boldsymbol{\varrho}_j)\cdot\boldsymbol{s}_0 = 0 \quad \text{and} \quad \boldsymbol{R}_0 = \boldsymbol{\varrho}_0 = -\boldsymbol{r}_0.$$

Application of the stationary phase method to the second integral in (2.9) demonstrates that the estimated contribution at the shadow boundary coming from the continuous longitudinal wave is at most of the order of $k^{-\frac{1}{2}}$. On the other hand, the contribution from the integral (3.8),

$$i\,\frac{\sqrt{(r+\varrho)-R}}{2r\varrho(r\varrho+\boldsymbol{r}\cdot\boldsymbol{\varrho})}\,\frac{e^{i\left(k_1R\pm\frac{\pi}{4}\right)}}{\sqrt{|\xi_j''|}}\left(ik_1-\frac{1}{R}\right)(\boldsymbol{r}\times\boldsymbol{\varrho})\cdot\boldsymbol{s}_0F(v) \tag{3.9}$$

is of the order of k. Hence, at the shadow boundary the decisive terms is that containing the function $\boldsymbol{W}_0$, which among other things shows the correctness of the choice of the potential (2.6).

In the immediate neighbourhood of the shadow boundary the second term in (3.8) may be neglected as

$$(r+\varrho_0-R)\to 0 \quad \text{and} \quad \varrho_0-\boldsymbol{R}_0\to 0;$$

also, $\int_{+\infty}^{k_1} F(v)dk_1'$ also tends to zero when $k\to+\infty$.

We shall also make use of some other, known identities valid at the shadow boundary, *viz.*,

$$\frac{1}{\sqrt{\xi''}}=\frac{\sqrt{r\varrho}}{\sqrt{R\sin^2(\varrho ds)}}$$

$$\frac{(\boldsymbol{r}\times\boldsymbol{\varrho})\cdot\boldsymbol{s}_0\sqrt{(r+\varrho)-R}}{(r\varrho+\boldsymbol{r}\cdot\boldsymbol{\varrho})r\varrho}=\frac{2}{r\varrho}\sqrt{\frac{r\varrho}{r+\varrho+R}}\sin(\varrho ds)\sin\tfrac{1}{2}(r,\varrho)\cos\alpha$$

with $\cos\alpha=+1$ in the light cone, and $\cos\alpha=-1$ in the geometrical shadow region.

With all this taken into account, the approximate formula for the diffracted wave near the shadow boundary has the form

$$\boldsymbol{u}^{(D)}|_{\text{sh.bound.}}\approx\left(ik_1-\frac{1}{R}\right)\frac{e^{i\left(k_1R\pm\frac{\pi}{4}\right)}}{\sqrt{2}\,R}\cos\alpha\int_{+\infty}^{\sqrt{\frac{k_1}{2\pi}(r+\varrho-R)}}e^{i\frac{\pi}{2}v^2}dv\boldsymbol{R}_0+\sum_l \boldsymbol{u}_{tr}|_l \tag{3.10}$$

where

$$\boldsymbol{u}_{tr}|_l\approx\sqrt{\frac{k_2}{8\pi|\vartheta_l''|}}\frac{e^{i\left(k_2\vartheta_l\pm\frac{\pi}{4}\right)}}{r\varrho}\left\{(\boldsymbol{r}_0\times\boldsymbol{s}_0)+\frac{2}{\kappa}\boldsymbol{r}_0\times[\boldsymbol{r}_0\times(\boldsymbol{s}_0\times\boldsymbol{\varrho}_0)]\right\}\Big|_l. \tag{3.10a}$$

In formula (3.10) the sum runs over the contributions from the critical points Q_l on the edge for which the function of arc ϑ accepts an extreme value. The form of (3.10) shows that the longitudinal diffracted wave undergoes a jump of

$$\left(ik_1-\frac{1}{R}\right)\frac{e^{ik_1R}}{R}\boldsymbol{R}_0$$

at the shadow boundary, which compensates the jump of the incident wave. From the extremum condition $\vartheta'=0$ it follows that the elementary transverse waves diverge in first approximation over the same half-cone with its apex at point Q_l. The longitudinal wave emerging from the element ds containing the same active point Q_l propagates over the surface of a cone having a smaller angle than the cone for the transverse waves emerging from the same element ds. This kind of conclusion was arrived at earlier by Gniadek

(1969) on the basis of the tensor potential in the form (2.3). The directions r_l of the elementary transverse waves leaving point Q_l form a circular cone which comprises the cone of the longitudinal wave coming from Q_l (Fig. 1). The direction of the incident wave ϱ_l and the direction r_l cannot be parallel except in the case of the half-plane when the source L and the observation P both lie on a plane perpendicular to the edge of the half-plane.

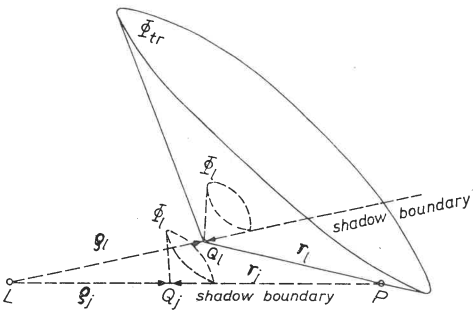

Fig. 1. Formation of the elementary diffracted wave; Φ_l — conical surfaces of the elementary longitudinal wave, Φ_{tr} — conical surface of the elementary transverse wave, Q_l, Q_j — critical points, L — source of longitudinal wave, P — observation point

Then, the points Q_j and Q_l overlap and both cones degenerate into a half-plane. Both transverse waves now have the same polarization, perpendicular to the diffracting edge, and formula (3.10) reduces considerably to

$$\boldsymbol{u}^{(D)}_{|j=l} \approx \left\{\left(ik_1 - \frac{1}{R}\right)\frac{e^{i\left(k_1R\pm\frac{\pi}{4}\right)}}{\sqrt{2}\,R}\cos\alpha \int\limits_{+\infty}^{\sqrt{\frac{k_1}{2\pi}(r+\varrho-R)}} e^{i\frac{\pi}{2}v^2}dv\boldsymbol{R}_0 + \right.$$

$$\left. + i\sqrt{\frac{k_2}{8\pi|\vartheta''|}}\,\frac{e^{i\left(k_2\vartheta\pm\frac{\pi}{4}\right)}}{r\varrho}\,\frac{\kappa+2}{\kappa}(\boldsymbol{r}_0\times\boldsymbol{s}_0)\right\}_{j=l}. \tag{3.10b}$$

We may now give the result of calculating (3.4) by the stationary phase method in the immediate vicinity of the shadow boundary. The first integral is reduced in essence to the scalar diffracted wave (Rubinowicz 1966, Sec. V) multiplied by $\boldsymbol{r}_0$. The second integral in (3.4) vanishes because $\boldsymbol{r}_0 || \boldsymbol{\varrho}_0$. Calculation of the integrals representing the transverse waves gives the same results as the corresponding terms in (3.10).

Hence, we immediately get the diffracted wave calculated from (3.4),

$$\boldsymbol{u}^{(D)}_{|\text{sh. bound}} \approx ik_1\frac{e^{i\left(k_1R\pm\frac{\pi}{4}\right)}}{\sqrt{2}\,R}\cos\alpha F(v)\boldsymbol{R}_0|_j + \sum_l \boldsymbol{u}_{tr}|_l \tag{3.10c}$$

as a special approximation of (3.10), in which $\boldsymbol{u}_{tr}|_l$ is given by (3.10a).

b) The diffracted wave at observation points far from the shadow boundary

For this region of observation we apply the stationary phase method directly to formula (3.4). The contribution to the diffracted wave from the critical points Q_j and Q_l is found to be given by

$$u^{(D)}_{|j,l} \approx i\sqrt{\frac{k_1}{8\pi|\zeta''|}}\,\frac{e^{i\left(k_1\zeta_j \pm \frac{\pi}{4}\right)}}{r\varrho}\left(\frac{1}{1+\cos(r,\varrho)}+\frac{2}{\kappa^2}\right)[(r_0\times\varrho_0)\cdot s_0]r_0|_j +$$

$$+\,i\sqrt{\frac{k_2}{8\pi|\vartheta''|}}\,\frac{e^{i\left(k_2\vartheta \pm \frac{\pi}{4}\right)}}{r\varrho}\left\{(r_0\times s_0)+\frac{2}{\kappa}r_0\times[r_0\times(s_0\times\varrho_0)]\right\}\Big|_l. \tag{3.11}$$

We consider the case of a half-plane. If s_0 defines the direction of the edge of the half-plane, then (3.11) represents the total diffracted wave received by the observer: from the point Q_j a longitudinal wave and from the point Q_l a transverse wave. The distance between the points Q_j and Q_l, and the angle between the directions incidence of the waves, are easily calculated functions of κ. The case when the points P and L lie in a plane perpendicular to the edge is interesting. Then, because

$$(r_0\times\varrho_0)\,||s_0 \quad\text{and}\quad r_0\times[r_0\times s_0\times\varrho_0)] = (r_0\times s_0)\cos(r,\varrho)$$

we have

$$u^{(D)}_{|j=l} \approx i\sqrt{\frac{k_1}{8\pi|\zeta''|}}\,\frac{e^{i\left(k_1\varrho \pm \frac{\pi}{4}\right)}}{r\varrho}\left(\frac{1}{1+\cos(r,\varrho)}+\frac{2}{\kappa^2}\right)\sin(r,\varrho)r_0 +$$

$$+\sqrt{\frac{k_2}{2\pi|\vartheta_l''|}}\,\frac{e^{i\left(k_2\vartheta \pm \frac{\pi}{2}\right)}}{r\varrho}\,\frac{2}{\kappa}(\cos\beta+\cos(r,\varrho))\,(s_0\times r_0)\ldots \tag{3.11a}$$

where $\beta = \arccos\kappa/2$.

It is seen from Eq. (3.11a) that at a fixed ϱ_0 the amplitude of the transverse wave takes on values nearly equal to zero for certain directions of observation ($\beta = \sphericalangle(r,\varrho)\pm\pi$). Hence, there is such a direction of observation for which the diffracted wave is in practice observed only as a longitudinal wave.

4. *Conclusion*

With the tendency of simplifying and improving the legibility of formulae for the diffracted wave use was made in this work of the exact Young approach to the formation of the diffracted wave. From the experimental point of view formulae (2.8) and (3.6) are equivalent. Notwithstanding, formula (2.8) and the respective subsequent forms of the diffracted wave are the outcome of the Heuristic role of Young's idea in the search for simple descriptions of physical phenomena. Along these same lines, it is possible to obtain simplifications and facilities in the interpretation of the formula for the diffracted wave already known in the literature in the case when the incident elastic wave is a transverse spherical wave (Gniadek 1969). This problem will be dealt with separately in a forthcoming paper.

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APPENDIX 1

Calculation of $(\hat{\Gamma}_1 \times \boldsymbol{u})\, \boldsymbol{s}_0$

First, let us note that the vectors $\boldsymbol{u}$ and $\boldsymbol{s}_0$ are independent of the coordinates of the observation point P. Using the form (2.3a) and the identity

$$\nabla \operatorname{div} (\boldsymbol{\delta}^{(i)} f_1) = (\boldsymbol{\delta}^{(i)} \cdot \nabla) \nabla f_1$$

we make the transformation

$$(\hat{\Gamma}_1 \times \boldsymbol{u})_{i,j} = -\frac{1}{\omega^2} [(\boldsymbol{\delta}^{(i)} \cdot \nabla_P) \nabla_P f_1 \times \boldsymbol{u}]_j.$$

By virtue of the identity

$$(\boldsymbol{c} \cdot \nabla)(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \times (\boldsymbol{c} \cdot \nabla)\, \boldsymbol{b} - \boldsymbol{b} \times (\boldsymbol{c} \cdot \nabla) \boldsymbol{a}$$

we put the tensor $\hat{\Gamma}_1 \times \boldsymbol{u}$ in the form of a tensor whose rows for $i = 1, 2, 3$ are the respective components of the vectors $-\frac{1}{\omega^2} (\boldsymbol{\delta}^{(i)} \cdot \nabla_P)(\nabla_P f_1 \times \boldsymbol{u})$. Thence,

$$(\hat{\Gamma}_1 \times \boldsymbol{u}) \boldsymbol{s}_0 = -\frac{1}{\omega^2} \nabla_P [\boldsymbol{s}_0 \cdot (\nabla_P f_1 \times \boldsymbol{u})]. \tag{A1.1}$$

Next, we transform the gradient in the following way:

$$\nabla_P [\boldsymbol{s}_0 \cdot (\nabla_P f_1 \times \boldsymbol{u})] = -\nabla_P [\nabla_P f_1 \cdot (\boldsymbol{s}_0 \times \boldsymbol{u})] = \nabla_P \left[\frac{1}{r} \frac{\partial}{\partial r} f_1 (\boldsymbol{s}_0 \times \boldsymbol{u}) \cdot \boldsymbol{r} \right] =$$

$$= -\frac{1}{r} \frac{\partial}{\partial r} f_1 (\boldsymbol{s}_0 \times \boldsymbol{u}) + [\boldsymbol{r} \cdot (\boldsymbol{s}_0 \times \boldsymbol{u})] \nabla_P \left(\frac{1}{r} \frac{\partial}{\partial r} f_1 \right). \tag{A1.2}$$

With use of the identity

$$-k_1^2 f_1 = \operatorname{div} \nabla f_1 = \operatorname{div} \boldsymbol{r}_0 \frac{\partial}{\partial r} f_1 = \frac{2}{r} \frac{\partial}{\partial r} f_1 + \frac{\partial^2}{\partial r^2} f_1$$

we have

$$\nabla_P \left(\frac{1}{r} \frac{\partial}{\partial r} f_1 \right) = \frac{1}{r} \left(\frac{3}{r} \frac{\partial}{\partial r} f_1 + k_1^2 f_1 \right) \boldsymbol{r}_0.$$

Formula (A1.2) now takes the form

$$\nabla_P[s_0 \cdot (\nabla_P f_1 \times u)] = -\frac{1}{r}\frac{\partial}{\partial r} f_1(s_0 \times u) + \left(\frac{3}{r}\frac{\partial}{\partial r} f_1 + k_1^2 f_1\right)[(s_0 \times u) \cdot r_0]r_0.$$

Whence, after substituting $u = \nabla_Q f(\varrho)$ and performing the rewritten differentiation operations, we get

$$(\hat{\Gamma}_1 \times u)s_0 = \frac{1}{\omega^2} f_1 f(\varrho)\,[A_1(s_0 \times \varrho_0) - B_1[(s_0 \times \varrho_0) \cdot r_0]r_0] \qquad \text{(A1.3)}$$

where A_1 and B_1 are coefficients given by Eqs (2.7b) and (2.7b′).

Formulae (A1.1) and (A1.3) are the forms of the vector $(\hat{\Gamma}_1 \times u)s_0$ appearing in Eq. (2.7).

APPENDIX 2

Calculation of $(\hat{\Gamma}_2 \times u)s_0$

Proceeding similarly as in Appendix 1 we get by virtue of (2.3b) the expression

$$(\hat{\Gamma}_2 \times u)s_0 = \frac{1}{\omega^2}\{\nabla_P[s_0 \cdot (\nabla_P f_2 \times u)] + k_2^2 f_2(u \times s_0)\}. \qquad \text{(A2.1)}$$

Making use of the fact that f_2 satisfies the equation of vibrations, we get the following transformation:

$$f_2(u \times s_0) = -\frac{1}{k_2^2}(u \times s_0)\,\mathrm{div}_P \nabla_P f_2 =$$

$$= -\frac{1}{k_2^2}\{\mathrm{curl}_P\,[(u \times s_0) \times \nabla_P f_2] + [(u \times s_0) \cdot \nabla_P]\nabla_P f_2\}. \qquad \text{(A2.2)}$$

After placing (A2.2) into (A2.1) and considering that

$$[(u \times s_0) \cdot \nabla_P] = -\frac{1}{\omega^2}\,\mathrm{curl}_P\,[(u \times s_0) \times \nabla_P f_2]$$

we get the first identity (2.7a),

$$(\hat{\Gamma}_2 \times u)s_0 = -\frac{1}{\omega^2}\,\mathrm{curl}_P\,[(u \times s_0) \times \nabla_P f_2]. \qquad \text{(A2.3)}$$

We get the second identity (2.7a) from (A2.3) in the following manner. After applying the formula for the double vector product, performing the curl operation and the following gradient operations,

$$\nabla_P(\nabla_P f_2 \cdot u) = -\nabla_P\left[\left(\frac{1}{r}\frac{\partial}{\partial r} f_2\right) r \cdot u\right] =$$

$$= -\omega_\varrho \frac{\partial}{\partial r} f_2 + (r_0 \cdot u) r_0 \frac{\partial^2}{\partial r^2} f_2$$

and

$$\nabla_P(\nabla_P f_2 \cdot s_0) = -\omega_s \frac{\partial}{\partial r} f_2$$

where

$$\omega_\varrho = \frac{1}{r}[(r_0 \cdot u)r_0 - u]$$

$$\omega_s = \frac{1}{r}[(r_0 \cdot s_0)r_0 - s_0]$$

we get the converted form of (A2.3)

$$(\hat{\Gamma}_2 \times u)s_0 = -\frac{1}{\omega^2}\frac{\partial}{\partial r} f_2(s_0 \times \omega_\varrho) + (s_0 \times r_0)(r_0 \cdot u)\frac{\partial^2}{\partial r^2} f_2 +$$

$$+(u \times \omega_s)\frac{\partial}{\partial r} f_2 - (u \times r_0)(r_0 \cdot s_0)\frac{\partial^2}{\partial r^2} f_2 =$$

$$= \frac{1}{\omega^2 r}\{2(s_0 \times u) + [u(s_0 \cdot r_0) - (r_0 \cdot u)s_0] \times r_0\}\frac{\partial}{\partial r} f_2 +$$

$$+ \frac{\partial^2}{\partial r^2} f_2[s_0(r_0 \cdot u) - u(r_0 \cdot s_0)] \times r_0. \tag{A2.4}$$

We express the member in the square brackets of the last equality (A2.4) as a triple vector product, consider the identity

$$\frac{\partial^2}{\partial r^2} f_2 = -\frac{2}{r}\frac{\partial}{\partial r} f_2 - k_2^2 f_2$$

and get

$$(\hat{\Gamma}_2 \times u)s_0 = \frac{1}{\omega^2}\left\{\frac{2}{r}\frac{\partial}{\partial r} f_2(s_0 \times u) + \left(\frac{3}{r}\frac{\partial}{\partial r} f_2 + k_2^2 f_2\right)[r_0 \times (u \times s_0)] \times r_0\right\}. \tag{A2.5}$$

After substituting $u = \nabla_Q f(\varrho)$ and performing the operation of differentiation with respect to r and ϱ, (A2.5) becomes after arranging the various terms the second identity (2.7a).

APPENDIX 3

Proof that the elementary wave given by formula (2.8) fulfills the equation of motion (2.1)

a) We shall show that the various members of the longitudinal wave fulfill the equation of motion.

In the case of the member $\nabla_P(s_0 \cdot W_0)$ the proof is instantaneous, once it is noticed that $W_0 \cdot s_0$ is an elementary diffracted wave in the scalar case. This wave satisfies the

Helmholtz equation of vibrations, whence

$$\nabla_P \operatorname{div}_P \nabla_P(\boldsymbol{W}_0 \cdot \boldsymbol{s}_0) = -\nabla_P k_1^2(\boldsymbol{W}_0 \cdot \boldsymbol{s}_0).$$

In the case of the other member of the longitudinal wave we find on the basis of the identity

$$\nabla_P[\boldsymbol{s}_0(\nabla_P f_1 \times \boldsymbol{u})] = \operatorname{curl}_P[(\boldsymbol{s}_0 \times \boldsymbol{u}) \times \nabla_P f_1] - k_1^2 f_1(\boldsymbol{s}_0 \times \boldsymbol{u})$$

the result

$$\nabla_P \operatorname{div}_P \nabla_P[\boldsymbol{s}_0(\nabla_P f_1 \times \boldsymbol{u})] = -k_1^2 \nabla_P[\nabla_P f_1 \cdot (\boldsymbol{s}_0 \times \boldsymbol{u})].$$

Hence, the member $\nabla_P[\boldsymbol{s}_0 \cdot (\nabla_P f_1 \times \boldsymbol{u})]$ fulfills the equation of motion.

b) To prove that both transverse waves fulfill the equation of motion it suffices to show that both members of the transverse wave satisfy the equation

$$\operatorname{curl}_P \operatorname{curl}_P \boldsymbol{w} = k_2^2 \boldsymbol{w}$$

where $\boldsymbol{w}$ is a sourceless vector.

In the case of the first transverse wave the integrand is converted into the form $-\boldsymbol{s}_0 \times \nabla_P f_2 f(\varrho)$. After using the formula for the curl of a vector product we get

$$\operatorname{curl}_P \operatorname{curl}_P \operatorname{curl}_P (f(\varrho) f_2 \boldsymbol{s}_0) = -f(\varrho) \operatorname{curl}_P [\boldsymbol{s}_0 \operatorname{div}_P \nabla_P f_2 +$$
$$+(\boldsymbol{s}_0 \cdot \nabla_P)\nabla_P f_2] = k_2^2 f(\varrho) \operatorname{curl} \boldsymbol{s}_0 f_2$$

which proves the thesis.

In the case of the last integral in (2.8) the proof is analogous.

APPENDIX 4

Proof that the vector $\hat{G}\boldsymbol{s}_0 = \boldsymbol{G}$ does not fulfill the equation of motion (2.1)

The vector $\boldsymbol{G}$ does not contain the parameter k_2, but only the parameter k_1. Hence, the condition which has to be satisfied by $\boldsymbol{G}$ for Eq. (2.1) to be complied with is the disappearance of the double curl of $\boldsymbol{G}$.

We show that:

a) $\operatorname{div}_P \boldsymbol{G} = 0$

b) $\operatorname{curl}_P \operatorname{curl}_P \boldsymbol{G} \neq 0.$

The product of the tensor $\hat{G}$ (3.4) times the vector $\boldsymbol{s}_0$ can be presented as

$$\hat{G}\boldsymbol{s}_0 = \boldsymbol{G} = (\boldsymbol{s}_0 \cdot \nabla_Q)\boldsymbol{W}_0 = \boldsymbol{i}\left[s_x \frac{\partial}{\partial x_Q} W_x + s_y \frac{\partial}{\partial y_Q} W_x + s_z \frac{\partial}{\partial z_Q} W_x\right] +$$
$$+\boldsymbol{j}\left[s_x \frac{\partial}{\partial x_Q} W_y + s_y \frac{\partial}{\partial y_Q} W_y + s_z \frac{\partial}{\partial z_Q} W_y\right] +$$
$$+\boldsymbol{k}\left[s_x \frac{\partial}{\partial x_Q} W_z + s_y \frac{\partial}{\partial y_Q} W_z + s_z \frac{\partial}{\partial z_Q} W_z\right]. \qquad \text{(A4.1)}$$

Performing the divergence operation on (A4.1) yields

$$\operatorname{div}_P \hat{G}s_0 = \frac{\partial}{\partial x_P}\left[s_x \frac{\partial}{\partial x_Q} W_x + s_y \frac{\partial}{\partial y_Q} W_x + s_z \frac{\partial}{\partial z_Q} W_x\right] +$$

$$+ \frac{\partial}{\partial y_P}\left[s_x \frac{\partial}{\partial x_Q} W_y + \dots\dots\dots\right] +$$

$$+ \frac{\partial}{\partial z_P}\left[s_x \frac{\partial}{\partial x_Q} W_z + \dots\dots\dots\right].$$

By altering the succession of differentiation and grouping the terms appropriately it can be seen that $\operatorname{div}_P \boldsymbol{G}$ is a sum of three divergences of the sourceless vector $\boldsymbol{W}_0$,

$$\operatorname{div}_P (\hat{G}s_0) = s_x \frac{\partial}{\partial x_Q} \operatorname{div}_P \boldsymbol{W}_0 + s_y \frac{\partial}{\partial y_Q} \operatorname{div}_P \boldsymbol{W}_0 + s_z \frac{\partial}{\partial z_Q} \operatorname{div}_P \boldsymbol{W}_0 = 0.$$

The outcome of the $\operatorname{curl}_P \operatorname{curl}_P$ operation on vector $\boldsymbol{G}$ is a third-order polynomial of the variable k_1. It suffices to show that the coefficient standing at k_1 in any power is not equal to zero. We calculate the coefficient at k_1^3. We perform the successive differentiations, leaving only the term having the coefficient at the highest appearing power of k_1:

$$\boldsymbol{G} = -\nabla_L(\boldsymbol{W}_0 \cdot \boldsymbol{s}_0) + \boldsymbol{s}_0 \times \operatorname{curl}_L \boldsymbol{W}_0 + \boldsymbol{s}_0 \times \operatorname{curl}_P \boldsymbol{W}_0 - \nabla_P(\boldsymbol{W}_0 \cdot \boldsymbol{s}_0).$$

Making use of the formula for the gradient of the scalar product of vectors for the proper pair of members yields

$$\boldsymbol{G} = -(\boldsymbol{s}_0 \cdot \nabla_P)\boldsymbol{W}_0 - (\boldsymbol{s}_0 \cdot \nabla_L)\boldsymbol{W}_0 = -(\boldsymbol{r} \times \boldsymbol{\varrho})\,[\boldsymbol{s}_0 \cdot (\nabla_P g + \nabla_L g)]$$

where

$$g = -\frac{1}{4\pi} f_1 f(\varrho) \frac{1}{r\varrho + \boldsymbol{r} \cdot \boldsymbol{\varrho}}; \qquad \nabla g \approx -\frac{1}{4\pi} \frac{1}{r\varrho + \boldsymbol{r} \cdot \boldsymbol{\varrho}} \nabla_P f_1 f(\varrho).$$

In first approximation we have

$$\boldsymbol{G} \approx ik_1(\boldsymbol{r} \times \boldsymbol{\varrho}) g[\boldsymbol{s}_0 \cdot (\boldsymbol{r}_0 + \boldsymbol{\varrho}_0)]. \tag{A4.2}$$

Application of the formula for the curl,

$$\operatorname{curl} \varphi \boldsymbol{a} = \varphi \operatorname{curl} \boldsymbol{a} - \boldsymbol{a} \times \nabla \varphi$$

twice to (A4.2) leads in first approximation to the result

$$\operatorname{curl}_P \operatorname{curl}_P \boldsymbol{G} \approx -k_1^3 g[\boldsymbol{s}_0 \cdot (\boldsymbol{r}_0 + \boldsymbol{\varrho}_0)]\,[(\boldsymbol{r} \times \boldsymbol{\varrho}) \times \boldsymbol{r}_0] \times \boldsymbol{r}_0 \neq 0$$

thus proving b).

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