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SOME SIMPLE MODELS OF ANISOTROPIC FERMI LIQUIDS. II

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The effects resulting from anisotropic effective interaction of an electron Fermi liquid are studied. The results obtained by the author previously are applied in an investigation of the basic properties of an anisotropic Fermi liquid with a spherical Fermi surface: the specification of the point group of the crystal is rather irrelevant here. The relation between the amplitudes of the effective interaction and the scattering amplitude, the conditions of isotropy of the effective mass, and the derivative of the Fermi momentum with respect to the external hydrostatic pressure, are found. Moreover, the stability conditions, other constraints and the sum rule are discussed together with questions associated with electron-phonon interaction.

In the Landau-Silin approach [1, 2] (*cf.* also [3, 4] and the monograph [5]) the electron properties of metals are determined by the one-quasiparticle spectrum, the effective interaction of quasiparticles and the kernel appearing in the linearized collisions integral, provided the low-temperature long-wavelength response of the system is studied. In a majority of papers on this topic the effective interaction of quasiparticles as well as collisions kernel appear either in a quite general form or in the isotropic one (*i. e.* only with the dependence on the cosine of the angle between momentum vectors on the Fermi surface). There are some exceptions from the above rule. Recently, Rice [6] considered the anisotropy of the effective interaction induced by phonon exchange; on the other hand, systems which can be obtained from isotropic ones by a homogeneous three-axial dilatation of the momentum space [7, 8] have also been considered. Moreover, the problem of the most general function invariant under some point group describing, *e. g.*, the effective interaction, was discussed by us in the paper [9]. The result of this paper will be applied by us to investigate the simplest physical consequences of more general effective interactions, such as static properties, stability conditions and other constraints, the relation between the effective interaction and the scattering amplitude of quasiparticles (see, for instance, [5] and [10]), and the sum rule [6, 11]. It is clear that the group-theoretical analysis of the most general effective interaction is independent of the particular form of the Fermi surface as compared with a more physical approach. On the other hand, consider-

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ation of non-spherical Fermi surfaces from the physical point of view leads to serious computational difficulties even for isotropic interactions. Hence, we confine our reasoning to spherical Fermi surfaces, *i. e.* to alkali metals. The effects of anisotropy discussed by us are important for monocrystals only. Moreover, our considerations rather treat the problem of the anisotropy induced by bands when the effects are probably small (*cf.* [6]). Our discussion is restricted to the collisionless response of the system, when the form of collisions kernel is unimportant.

If the spin-orbit coupling is unimportant, then the effective interaction (depending on two momenta $\boldsymbol{p}$, $\boldsymbol{p}'$ and four spin indices) can be represented as

$$\hat{f}(\boldsymbol{p}, \boldsymbol{p}') = f_{\text{direct}}(\boldsymbol{p}, \boldsymbol{p}') + (\boldsymbol{\sigma}\boldsymbol{\sigma}') f_{\text{exchange}}(\boldsymbol{p}, \boldsymbol{p}') \tag{1}$$

where both f are symmetric with respect to interchange of $\boldsymbol{p}$ and $\boldsymbol{p}'$, and $\boldsymbol{\sigma}$ denote vectors of Pauli matrices. If we assume that (1) is invariant under some point group then, according to [9], both functions (1) can be written as

$$f(\boldsymbol{p}, \boldsymbol{p}') = f(\hat{\boldsymbol{p}}\hat{\boldsymbol{p}}') + \sum_{ll'} \sum_{mm'} f_{ll'}^{mm'} O_1[Y_{lm}(\hat{\boldsymbol{p}}) Y_{l'm'}(\hat{\boldsymbol{p}}')], \tag{2}$$

where O_1 denotes the projection operator of the trivial representation of the group considered and the summation over (lm) runs over some set necessary and sufficient to obtain all invariants specific to this group, $\hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$. Without any loss of generality one can assume that $f_{ll'}^{mm'}$ can be put equal to zero if the pair (lm) or $(l'm')$ does not appear in the set mentioned above and, on the other hand, that $f_{ll'}^{mm'}$ is symmetric under simultaneous transposition of (lm) and $(l'm')$ in order the function $f(\boldsymbol{p}, \boldsymbol{p}')$ to be symmetric. The functions (1) can be also expressed as

$$f(\boldsymbol{p}, \boldsymbol{p}') = \sum_{\lambda} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \sum_{a=1}^{n_\lambda(l)} \sum_{b=1}^{n_\lambda(l')} \{ f_{\lambda ll'}^{ab} [\sum_{r=1}^{d_\lambda} K_{lr}^{\bar{\lambda}a}(\hat{\boldsymbol{p}}) K_{l'r}^{\lambda b}(\hat{\boldsymbol{p}}')] \}. \tag{3}$$

Here, $K_{lr}^{\lambda a}(\hat{\boldsymbol{p}})$ is a linear combination of the functions $Y_{lm}(\hat{\boldsymbol{p}})$, $|m| \leqslant l$, transforming as the r-th row of the λ-th irreducible representation of dimension d_λ of the considered point group. The summation over λ runs over all irreducible representations of this group, $\bar{\lambda}$ denotes the representation complex-conjugate to λ. Moreover, $n_\lambda(l)$ determines the multiplicity of the representation λ in the irreducible representation of the three-dimensional orthogonal group and the variables a, b describe the degeneration of crystal harmonics. Here, without any loss of generality one can assume that

$$K_{lr}^{\bar{\lambda}a}(\hat{\boldsymbol{p}}) = [K_{lr}^{\lambda a}(\hat{\boldsymbol{p}})]^*, \qquad \int \frac{d\Omega}{4\pi} K_{lr}^{\bar{\mu}a}(\hat{\boldsymbol{p}}) K_{l'r'}^{\lambda b}(\hat{\boldsymbol{p}}') = \delta_{\lambda\mu}\delta_{ab}\delta_{ll'}\delta_{rr'} \tag{4}$$

where the integration over $d\Omega$ runs over spherical angles (*cf.* [9]). As a result of the reality and symmetry of the functions (1) we have

$$[f_{\lambda ll'}^{ab}]^* = f_{\bar{\lambda}l'l}^{ba}, \qquad f_{\bar{\lambda}ll'}^{ab} = f_{\lambda l'l}^{ba}. \tag{5}$$

Let us now consider problems more closely related to physics. Here, all our conclusions would be much less definite if we would not consider spherical Fermi surfaces. Note that small deviations of the Fermi surface from a spherical form can be considered by the perturbational approach, but this is not our objective.

The dimensionless scattering amplitude of quasiparticles (g) is determined by the effective interaction (f) as

$$g = f - fg \tag{6}$$

where the product fg has an operatorial character with the average of the intermediate momentum over the Fermi surface (for spherical surfaces this is the integral over $d\Omega/4\pi$). In order to obtain the relation (6) we should define the dimensionless quantities by the dimensional ones as

$$\begin{Bmatrix} f(\boldsymbol{p}, \boldsymbol{p}') \\ g(\boldsymbol{p}, \boldsymbol{p}') \end{Bmatrix} = \frac{2S_F}{(2\pi)^3 (V_p V_{p'})^{\frac{1}{2}}} \begin{Bmatrix} F(\boldsymbol{p}, \boldsymbol{p}') \\ G(\boldsymbol{p}, \boldsymbol{p}') \end{Bmatrix}, \quad \hbar = 1, \tag{7}$$

where S_F denotes the area of the Fermi surface, V_p the velocity of quasiparticles on the Fermi surface, and F, G the dimensional effective interaction and scattering amplitude, respectively. In (6) there is no coupling between the direct part of f and the exchange part of g, and between f_{ex} and g_d. If we put $f = f_0 + \Delta f$, $g = g_0 + \Delta g$, where f_0, g_0 are isotropic and Δf, Δg are assumed to be small compared with f_0, g_0, and disregard the term $\Delta f \Delta g$ in (6), we find after some manipulations

$$\Delta g = (1 + f_0)^{-1} \Delta f (1 + f_0)^{-1}. \tag{8}$$

Let us denote the invariants specific for some group and obtained by the projection procedure from $Y_{lm}(\hat{\boldsymbol{p}})\, Y_{l'm'}(\hat{\boldsymbol{p}}')$ by $W_{ll'}^{mm'}(\hat{\boldsymbol{p}}, \hat{\boldsymbol{p}}')$. The Legendre amplitudes, Q_l, of an arbitrary function Q depending on $x = \hat{\boldsymbol{p}}\hat{\boldsymbol{p}}'$ are defined as follows:

$$Q(x) = \sum_{l=0} (2l+1) Q_l P_l(x). \tag{9}$$

From the addition theorem for spherical functions one finds

$$Q W_{ll'}^{mm'} = Q_l W_{ll'}^{mm'}, \qquad W_{ll'}^{mm'} Q = Q_{l'} W_{ll'}^{mm'}. \tag{10}$$

Taking into account that the Legendre amplitudes of the operator inverse to $(1 + f_0)$ will be $(1 + f_l)^{-1}$ and Eq. (2), one can write (8) as

$$\Delta g(\boldsymbol{p}, \boldsymbol{p}') = \sum_{ll'mm'} f_{ll'}^{mm'} [(1 + f_l)(1 + f_{l'})]^{-1} W_{ll'}^{mm'}(\boldsymbol{p}, \boldsymbol{p}'). \tag{11}$$

It should be noted that we can also find relations analogous to (10) and (11) if we describe f and g in terms of the invariants appearing in the formula (3) and separate the terms corresponding to their isotropic part.

In order to obtain the non-perturbational relation between the amplitudes g and f one has to take into account the products $W_{ll'}^{mm'} W_{nn'}^{kk'}$ as a result of the term $\Delta f \Delta g$. This product is proportional to $\delta_{l'n}$ and invariant under O_h or any other point group considered. Hence, for $l' = n$, this product can be expressed as a linear combination of the invariants $W_{ln'}$ with fixed l and n', and even $P_l(\hat{\boldsymbol{p}}\hat{\boldsymbol{p}}')$, but only for $l = n'$. If the last term is excluded for some l, then the relation between the l-th amplitudes f_0 and g_0 is the same as in the isotropic case. The appearance of the term proportional to P_l in the above expressions may seem suspect, but $W^{00}_{02k}W^{0\ 0}_{2k0}\sim P_0$ and (for invariants of the group O_h, O, T_d, T_h, T only) $W^{0m}_{12k+1}W^{m'\ \ 0}_{2k+11} = B^k_{mm'}P_1$ and $B^k_{mm'} \neq 0$ for some m and m'. On the other hand, this is strictly connected with necessary vanishing of the number of specific crystals invariants for $l = l' = 0$, whereas for $l = l' = 1$ this number vanishes only for the groups listed above. Taking into account (4) and (6) we can write the relations between expansion coefficients (3) of the effective interaction and the scattering amplitude. Denoting these coefficients by $f^{ab}_{\lambda ll'}$ and $g^{ab}_{\lambda ll'}$, respectively, we can write

$$g^{ab}_{\lambda ll'} = f^{ab}_{\lambda ll'} - \sum_{l''=0}^{\infty} \sum_{c=1}^{n_\lambda(l'')} f^{ac}_{\lambda ll''} g^{cb}_{\lambda l''l'}. \tag{12}$$

Now let us seek the answer to two questions:
1) when is the effective mass isotropic, and,
2) when is $\partial|\boldsymbol{p}_F|/\partial\mu$ isotropic, *i. e.* when does the Fermi surface remains spherical under external hydrostatic pressure? Let us apply the relations between reduced vertex functions in the "q" and "ω" limits [4]. These relations have a simple physical meaning in the phenomenological approach [5]. Taking into account that we are dealing with a spherical Fermi surface we can write, using the results of [4],

$$\tau^{aq}_{p} = \tau^{a\omega}_{p} - \frac{p_F^2}{\pi^2} \int \frac{d\Omega'}{4\pi} G(\boldsymbol{p}, \boldsymbol{p}') V^{-1}_{p'} \tau^{a\omega}_{p'}, \tag{13}$$

$$\tau^{a\omega}_{p} = \tau^{aq}_{p} + \frac{p_F^2}{\pi^2} \int \frac{d\Omega'}{4\pi} F(\boldsymbol{p}, \boldsymbol{p}') V^{-1}_{p'} \tau^{aq}_{p'}, \tag{14}$$

where now $|\boldsymbol{p}| = |\boldsymbol{p}'| = p_F$ and both relations (13) and (14) are equivalent because of the relation which relates G with F and which can be simply deduced from (6) and (7). Here, as a result of Ward's identities, we have

$$\tau^{aq}_{p} = \begin{cases} V_p \dfrac{\partial|\boldsymbol{p}_F|}{\partial\mu}, & a = 0 \\ V^a_p, & a = 1, 2, 3 \end{cases} \tag{15}$$

$$\tau^{a\omega}_{p} = \begin{cases} 1, & a = 0 \\ v^a_p \equiv p_a/m, & a = 1, 2, 3 \end{cases} \tag{16}$$

Note that the result (16) for $a > 0$ can be obtained only when the excitation energy without the contribution of the interelectron interaction is equal to $p^2/2m$ (m is called lattice or optical electron mass). On the other hand, F and G denote the spin-direct parts of the effective interaction and the scattering amplitude, respectively. Both these functions are invariant under the group considered and can be expanded into a series of the form (2) or (3). The functions f and g are invariant according to (7), provided the functions F, G are invariant and *vice versa*, but the relations between the expansion coefficients (2) or

(3) of the function F and f (or G and g) are very complicated if V_p is anisotropic. Moreover, if we express (13) and (14) by the functions f and g, then some quantities not having a simple physical meaning will appear (such as $V_p^{-\frac{1}{2}}\partial|\boldsymbol{p}_F|/\partial\mu$ and $V_p^{-\frac{1}{2}}V_p^a$). Hence, it is useful to discuss the above questions rather in terms of the expression (2) or (3) for the functions F and G. Taking into account that V_p^a/V_p as a unit vector normal to the Fermi surface is equal to p_a/p_F for spherical surfaces and putting $a > 0$ in formula (19), we find that the condition $F_{1,2k+1}^{m,m'} = F_{2k+1,1}^{m',\ m} = 0$ is equivalent to the isotropy of V_p. If this is fulfilled, then also $f_{1,2k+1}^{m,m'} = f_{2k+1,1}^{m',\ m} = 0$. Considering (13) under the above condition we obtain that the same amplitudes of the function G (or g) vanish and, as a result of equivalence of (13) and (14), both of these conditions are equivalent. Moreover, if all amplitudes $F_{1,2k+1} = F_{2k+1,1}$ are equal zero, then the relations between the Legendre amplitudes of the isotropic part of the functions f and g for $l = 1$ remains unaltered in comparison to the isotropic case (*i. e.* for $\Delta f = \Delta g = 0$) as well as the Landau relation for the effective mass. Analogously, for isotropic V_p, we find from (13) and (14) for $a = 0$ that if $f_{0,2k}^{0,m} = f_{2k,0}^{m,0}$ vanish (this is equivalent to $g_{0,2k}^{0,m} = g_{2k,0}^{m,\ 0} = 0$), then $\partial|\boldsymbol{p}_F|/\partial\mu$ is isotropic and *vice versa*. In this case also the relation between zeroth Legendre amplitudes of the functions f_0, g_0 has the same form as for isotropic systems. On the other hand, we can also prove that if $\partial|\boldsymbol{p}_F|/\partial\mu$ is isotropic and $F_{0,2k}^{0,m} = F_{2k,0}^{m,0} = 0$ then V_p is isotropic. Using the above results one can easily prove that if $F_{0,2k}^{0,m} = F_{1,2k+1}^{r,s} = 0$, then the expression $\partial N/\partial\mu$ has the same form as for the isotropic system. Moreover, if $F_{1,2k+1}^{r,s} = 0$ and the coefficient near the invariants $W_{0,2k}^{0,m}$ vanish in the spin-exchange part of the effective interaction, then the expression for the spin susceptibility also remains unaltered. The above conclusions can be simply expressed also in terms of the expansion coefficients $F_{\lambda ll'}^{ab}$, the function F (*cf.* (3)). It is clear that the relations $F_{0,2k}^{0,m} = 0$ are equivalent to $F_{\lambda 0,2k}^{ab} = 0$, $k > 0$, whereas the relations $F_{1,2k+1}^{m,m'} = 0$ are equivalent to $F_{\lambda 1,2k+1}^{ab} = 0$, $k > 0$.

Let us pass to the problem of stability conditions. They will be obtained here according to the scheme proposed by Leggett [12]. After making rather simple modification of the original approach [12] we find that the stability conditions are equivalent to the single inequality

$$\langle U^{+}(1-g)\,U\rangle > 0. \tag{17}$$

Here, g is defined by (6), the product has an operatorial character with the average over spherical angles related with the intermediate $\boldsymbol{p}$-vector, and U is an arbitrary function depending upon $\vec{p} = \boldsymbol{p}/|\boldsymbol{p}|$ (or an arbitrary diagonal operator). On the other hand,

$$\langle\varphi(\hat{p}, \hat{p}')\rangle \equiv \int\frac{d\Omega}{4\pi}\int\frac{d\Omega'}{4\pi}\,\varphi(\hat{p}, \hat{p}'), \tag{18}$$

and we have separate inequalities (17) for the spin-direct and spin-exchange part of the dimensionless scattering amplitude of quasiparticles. Expanding U into a series of spherical functions we can see that (17) is equivalent to the positive definiteness of the Hermitean form determined by the matrix $\langle Y_{lm}^{+}(1-g)\,Y_{lm}\rangle$ (lm plays the role of a single index). If there exists a number l_0 such that $g_{ll'}$ vanishes unless both ll' are smaller than l_0, then the condition of the positive definiteness of the above Hermitean form is equivalent to the well-known determinant criterion. Taking into account that $g = g_0 + \Delta g$ and the addition theorem for spherical functions we can write this criterion as

$$\mathrm{Det}\,\{\delta_{ll'}\delta_{mm'}(1-g_l) - \sum_{kn} g_{ll'}^{kn}\langle Y_{lm}^{+} W_{ll'}^{kn} Y_{l'm'}\rangle\} > 0. \tag{19}$$

Here, l, l' and m, m' are taken for l, l' smaller or equal some number L, whereas $|m|$, $|m'| \leqslant k_0 \leqslant L < l_0$, and the inequality is fulfilled for all $L < l_0$ and all $k_0 \leqslant L$. It should be noted that g_l and $g_{ll'}^{kn}$ are suitable expansion coefficients of the dimensionless scattering amplitude which are defined accordingly with (2). As a result of all inequalities in the form (19) we obtain the inequalities in the same form, but with lm belonging to an arbitrary set of pairs of indices, provided that $l'm'$ goes over the same set. The inequalities (19) can be simplified if we treat the second term as a perturbation. Assuming in this case that the isotropic part is itself stable (*i. e.* that $(1-g_l) > 0$) and taking into account that Det $(E+\varepsilon) = 1+\mathrm{Tr}\varepsilon$, we find using (11) that the single inequality

$$1 - \sum_{lm}(1+f_l)^{-1}\sum_{kn} f_{ll}^{kn}\langle Y_{lm}^{+} W_{ll}^{kn} Y_{lm}\rangle > 0 \tag{20}$$

is equivalent to all inequalities (19), provided the summation over lm goes only over positive terms. It is clear that the above equivalence holds only in the perturbative approach. It should be emphasized that we have here the inequalities (19) separately for the spin-direct and the spin-exchange parts of the scattering amplitude. If we apply the expansion (3) of the scattering amplitude (the coefficients $g_{\lambda ll'}^{ab}$) and take the function U expressed by the series of the functions $K_{lr}^{\lambda a}$, we find, using (4) and (17), that

$$\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{a=1}^{n_\lambda(l)}\sum_{b=1}^{n_\lambda(l')} [(\delta_{ab}\delta_{ll'} - g_{\lambda ll'}^{ab})(u_{lr}^{\lambda a})^{*} u_{l'r}^{\lambda b}] > 0, \tag{21}$$

where $U_{lr}^{\lambda a}$ denote coefficients of the expansion of the function U into the series of $K_{lr}^{\lambda a}$. If $g_{\lambda ll'}^{ab}$ vanishes unless both ll' are smaller than some number, then the well-known determinant criterion is equivalent to (21).

Let us pass to the proof of the analogue of the Leggett inequality [12]. It is the result of the non-conservation of the spin current, whereas the ordinary current is conserved. After introducing a simple modification to the original approach [12] (*cf.* also [13]), we can write for cubic point groups (*i.e.* for O_h, O, T_d, T_h, T)

$$\frac{p_F^2}{\pi^2}\left[\langle V_p(\hat{p}\hat{k})^2\rangle + \frac{p_F^2}{6\pi^2}(F_{d1}+F_{e1})\right] \leqslant \frac{N}{m}, \tag{22}$$

where F_{d1} and F_{e1} denote respectively the first ($l = 1$) Legendre amplitudes of the isotropic part of the direct and exchange effective interaction, and $\hat{p} = p/|p|$, *etc.* The analogue of the inequality (22) can be simply written also for remaining point groups if we take into account other invariants of these groups in $D^1 \otimes D^1$.

Let us pass now to the anisotropy induced by phonons. According to the classical result of Migdal [14], we can consider the single-phonon exchange with very good accuracy up to $\sqrt{m/M}$ (M is ionic mass, but only in the scattering amplitude of quasiparticles).

Hence, the scattering amplitude is the sum of the interelectron part and the electron--phonon part. In this case it is better to describe the system in terms of the scattering amplitude of quasiparticles (*cf.* [6]). As a result of single-phonon exchange and the cubic invariance of the phonon spectrum we have

$$g_{ph}(\hat{p},\hat{p}') = g_{ph}(\hat{p}-\hat{p}') \equiv g_{ph}(|\hat{p}-\hat{p}'|,\hat{q}) = g_{ph,0}(|\hat{p}-\hat{p}'|^2)+ \\ + \sum_{k=2,\alpha} g^{\alpha}_{ph,2k}(|\hat{p}-\hat{p}'|^2)K^{\alpha}_{2k}(\hat{q}), \quad \hat{q} = (\hat{p}-\hat{p}')/|\hat{p}-\hat{p}'|, \tag{23}$$

where the summation over α goes over degenerate trivial cubic harmonics (they appear for $k \geqslant 6$) and $(\hat{p}-\hat{p}')^2 = 2(1-\hat{p}\hat{p}')$. Note that the limit $\hat{p}' \to \hat{p}$ (important, for example, in the sum rule) depends on the direction of the vector $\hat{p}-\hat{p}'$, provided there appear such $k \geqslant 2$ that $g_{ph,2k}(0) \neq 0$. On the other hand, if we take our formulae (2) and (3) for the function $g(\boldsymbol{p},\boldsymbol{p}')$, then should one perform the limiting transition $\hat{p}' \to \hat{p}$ term by term, we would find that there is no dependence on the direction $\hat{p}-\hat{p}'$. Hence, we obtain that if we expand even the function $K^{\alpha}_{2k}(\hat{q})$ into a series of $W^{m,m'}_{l,l'}(\hat{p},\hat{p}')$ we have to get an infinite series for which the limit at $\hat{p}' \to \hat{p}$ cannot be found term by term. This is true provided the above series is convergent for $\hat{p} \neq \hat{p}'$. Mathematically, it is rather difficult to find the constraints on the function $g(\hat{p}-\hat{p}')$ which define it as expandable into a convergent series of two-vector cubic invariants. The anisotropic terms in the series expansion (23) were estimated in the paper [6]; it was found that $g_{ph,0}(0) \gg g_{ph,2k}(0)$.

Now we consider the sum rule for our system, which is the result of the vanishing of the scattering amplitude of quasiparticles for equal spins and momenta, *i.e.* the result of the Pauli principle. According to [11] we have

$$\lim_{|\hat{p}-\hat{p}'|\to 0} G(\boldsymbol{p}\uparrow,\boldsymbol{p}'\uparrow) = -\left(V_p \frac{\partial|\boldsymbol{p}_F|}{\partial\mu}\right)^2 \left(\frac{\partial N}{\partial\mu}\right)^{-1} \times \\ \times \left[1 - \frac{N_a Z^2}{M}\left(\frac{\partial N}{\partial\mu}\right)^{-1} \sum_{\lambda} (\hat{q}\varepsilon_{\lambda}(\hat{q})/v_{\lambda}(\hat{q}))^2\right], \tag{24}$$

where N denotes the density of electrons, Z the valence, $N_a = N/Z$, whereas $v_{\lambda}(\hat{q})$ and $\varepsilon_{\lambda}(\hat{q})$ denote, respectively, the velocity and polarization of phonons at small wave-vector values. According to our previous conclusions, $G(\hat{p}\uparrow,\hat{p}'\uparrow)$ is the sum of the interelectron part and the electron-phonon part. There is no reason for the first of them to depend on the $\hat{q}$-vector, but the second one has to depend on this vector. Let us average the formula (24) over the directions of the vector $\hat{q}$ tangent to the Fermi surface at given point $\boldsymbol{p}$, i.e. over all vectors $\hat{q}$ perpendicular to the vector $\boldsymbol{p}$. The suitable formula for the average value has the form

$$\langle A(\hat{q})\rangle_{\hat{q}\perp\hat{p}} \equiv \int \frac{d\Omega_q}{2\pi}\,\delta(\hat{q}\hat{p})A(\hat{q}). \tag{25}$$

It can be easily verified that (25) averaged over spherical angles connected with the vector $\boldsymbol{p}$ simply gives the average of $A(\hat{q})$ over spherical angles connected with the vector $\hat{q}$ and, moreover, $\langle \text{const} \rangle_{\hat{q} \perp \hat{p}} = \text{const}$. Applying the above procedure to (24) we find that the average does not change the interelectron part and, therefore we have the problem of averaging the electron-phonon part. Taking into account that $P_l(0)$ vanishes for odd l and is equal to $(-1)^{l/2}\ (l-1)!!/l!!$ for even l, we can write

$$\delta(\hat{q}\hat{p}) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!} (2k+\tfrac{1}{2}) P_{2k}(\hat{q}\hat{p}), \tag{26}$$

where we assumed, quite formally, that $0!! = (-1)!! = 1$. Applying the addition theorem for spherical functions and using (25) and (26) we obtain

$$\langle Y_{lm}(\hat{q}) \rangle_{\hat{q} \perp \hat{p}} = \begin{cases} (-1)^{l/2}(l-1)!!/l!!, & (l - \text{even}) \\ 0 & , (l - \text{odd}). \end{cases} \tag{27}$$

Since $K^{\alpha}_{2k}(\hat{q})$ are some linear combinations of the functions $Y_{2k,m}$, the functions $K^{\alpha}_{l}(\hat{q})$ fulfill the same averaging formula. Hence,

$$\langle \lim_{|\hat{p}-\hat{p}'| \to 0} g_{ph}(\hat{p}, \hat{p}') \rangle_{\hat{q} \perp \hat{p}} = g_{ph,0}(0) +$$

$$+ \sum_{k=2,\alpha}^{\infty} g^{\alpha}_{ph,2k}(0)\, (-1)^k \frac{(2k-1)!!}{(2k)!!} K^{\alpha}_{2k}(\hat{p}) \tag{28}$$

and we can write the sum rule using (28) and suitably averaged (24). In this case it is rather difficult to express analytically the scattering amplitude in terms of the parameters of the effective interaction, mainly because of the appearance of the term $g_{ph}(\hat{p}-\hat{p}')$. One can simplify the averaged formula (24) by introducing here the expression for $\partial N/\partial\mu$, the dimensionless scattering amplitude, *etc.* It can be easily seen that all formulae concerning the anisotropy induced by phonons hold for any other point group with K^{α}_{2k} — the trivial representation of this group. Thus, K^{α}_{2} can also appear for point groups other than O_h, O, T, T_d, T_h. On the other hand, our considerations are valid only for $\boldsymbol{q} \perp \boldsymbol{p}$, *i.e.* for spherical Fermi surfaces, if we do not place additional restrictions on $\boldsymbol{p}$.

The above considerations and the analogous ones of paper [9] constitute a necessary step to the theory of measurable effects. Among them, the most significant effect is the effect of spin waves in an external magnetic field [15]. Hence, the theory of spin waves in monocrystals will be our query in the nearest future, even though they have hitherto been measured in polycrystalline samples. On the other hand, it follows from the peculiar properties of electron-phonon interaction that the formulation of the theory in terms of the scattering amplitude of quasiparticles is necessary in order to solve this problem.

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