# GENERALIZED IRREVERSIBLE THERMODYNAMICS AND ITS APPLICATION TO LASERS. PART II. THERMODYNAMICS OF A LASER

BY R. S. INGARDEN

Institute of Physics, Nicholas Copernicus University, Toruń*

*(Received January 27, 1972)*

On the base of the general theory developed in Part I, the statistical thermodynamics of laser operation is given. The method is general, but the concrete calculation was carried out for a certain simple model of a solid state laser. This model consists of $N$ two-level lasing atoms and one mode of an electromagnetic cavity in exact resonance with the atoms. The dynamical behaviour of the system is described by a usual Hamiltonian (the "inner" behaviour) and dissipation and pumping operators (the "outer" behaviour, the influence of environment) chosen in confirmity with the conditions given in Part I. A 6- and 11-temperature thermodynamics of the laser action is given, and its result is that in the asymptotic stationary state (time $t = \infty$) below the threshold of pumping the solution is ergodic (independent of the initial state), and above the threshold it is semi-ergodic (depends on 2 initial mean values, namely electric and magnetic signals). It is shown also that in the 11-temperature thermodynamics a second order energetic temperature of the electromagnetic field occurs, in conformity with the well-known theoretical and experimental results (Risken 1968). An analogy to the working of superconductors is indicated.

## 1. *Introduction*

Although the laser is such a recent device, the literature on it is enormous: in the first 10 years of its existence more than 10000 papers were published (*cf.* Haken 1970, p. 1). Even laser theory has very extensive literature and its review is a large undertaking as is shown by the mentioned Haken's article (Haken 1970) which actually is a book, a separate volume of *Handbuch der Physik* of 320 pages. Of course, also such an extensive review cannot be exhaustive, as is explicitly stated by its author who said on its first page: "we refer in the present article only to those papers which were used in its preparation or which are closely related to the special topics treated". Therefore, we do not try here to give a synthetic image of the existing theory of lasers as an introduction to our own contribution.

* Address: Instytut Fizyki, Uniwersytet M. Kopernika, Toruń, Grudziądzka 5, Poland.

We give only some short remarks considering the character of our contribution and its place in the existing theory. In the latter there are two branches: the microscopic theory and the phenomenological one, although there are also many papers of a mixed character. Because, however, of the very unconventional features of laser problems, and the urgent need of quick explanation of the experimental results, most theoretical papers use many "Ansatz'es" and "plausible assumptions" of a more or less phenomenological character, but without closer theoretical explanation or foundation. In contradistinction to this situation in the present paper we are much less interested in obtaining direct experimental results than in the theoretical explanation and foundation of the laser operation. Most physicists, especially experimentalists, think that already Einstein's theory of spontaneous and stimulated emission plus the concept of negative temperature (inversed population) explain without rest the laser action, but this is a crude oversimplification. First of all, the Einstein theory concerns only the equilibrium of a radiation field (the black body radiation) while laser radiation is very far from black body radiation. Secondly, as we shall show explicitly below, one temperature is not enough for the description of laser operation above the threshold (only below the threshold, but then the laser does not operate yet in the proper sense).

The main point of our theory is that it is a thermodynamical description of the laser action, thermodynamical in the sense explained in Part I of the present paper. Our thermodynamics is statistical and information-theoretical thermodynamics, it is also many--temperature (*cf.* Ingarden 1969) and irreversible, and therefore called generalized irreversible thermodynamics. From the point of view of Part I (which is a theoretical base for the present investigation) our laser theory is only an illustration. But from the point of view of laser theory, the subject of Part I is just what this theory needs for its later development: clear definition of all concepts, clean and self-consistent formulation of the method of calculation. Up to now we had too much details, too much clumsy and quick calculations for obtaining *per fas et nefas* the results which were mostly known boforehand from experimental investigations. The time has come for a quiet and dispassionate investigation and maintaining mathematical correctness. It is not a secret (since it is a rule in physical investigations) that foundations are built later than the upper stories. So most of our final results are well-known, but there are also new vistas and new results. Incidentally, we now know why some well-known results are correct (or approximately correct) while the assumptions under which they were obtained for the first time are positively false. Indeed, although our theory presented in Part I is very general, it is restrictive enough to exclude many of the current ideas and assumptions popular among physicists. Fortunately enough, most (although also not all) of the ideas and Ansatz'es of one of the leading schools in laser theory, that of Haken-Weidlich in Stuttgart (in which the author had good luck to work in 1966) appeared to be correct from the point of view of our theory.

The model of laser which we use here is a simple model of the solid-state laser with one mode of an electromagnetic cavity in current use by many authors (*cf. e.g.* Fleck 1966, 1966a, Fain and Khanin 1967). We use it here only for the sake of simplicity, since any other, more realistic, or one of another type of laser, may be also used as an example of our method. It is our hope that in future many different and more complicated models will be also treated by our method and that it will show its usefulness not only for school models.

The present investigation is a further development of one which was published in a short and imperfect summary (Ingarden 1971).

## 2. Dissipation operators

In Part I using the paper by Kossakowski (1972) we formulated the following generalized equation of motion for the density $\varrho(t)$ of a non-isolated system

$$\frac{d\varrho(t)}{dt} = L\varrho(t), \quad t \geqslant 0 \tag{2.1}$$

(Part I, Eqs (2.37), (2.39)), where we selected (for the sake of simplicity) the Schrödinger picture, and $t$ denotes time. $\varrho(t)$ operates in the Hilbert space of the system (since the Hilbert space is fixed, we do not use here its symbolic notation and reserve in Part II the letter $H$ for the Hamiltonian as below), while $L$, called the generalized Liouvillian (in short Liouvillian), is a linear operator (superoperator) acting in a dense set $D(L)$ in the Banach space $B_1$ defined in Part I, so

$$\varrho(t) \in D(L), \ \overline{D(L)} = B_1. \tag{2.2}$$

In general, we can decompose $L$ into two parts

$$L = L_0 + L_1, \tag{2.3}$$

where

$$L_0 = \frac{1}{i}[H, .], \ i.e. \ L_0\varrho = \frac{1}{i}[H, \varrho], \tag{2.4}$$

denotes the usual reversible Liouvillian describing the inner properties of the system and determined by the Hamiltonian $H$ of the system (being a self-adjoint, non necessarily bounded, linear operator, up to a constant positive definite, acting in a set $D(H)$ dense in the Hilbert space of the system), and $L_1$, called by us the dissipation operator, describes the influence of the surrounding on our system, and is the source of irreversible behaviour. In (2.4) we put the Planck constant $\hbar = 1$, $i^2 = -1$, and $[A, B] := AB - BA$ (the other brackets used in the following are: $(.,.)$ denotes the scalar product in the Hilbert space, $\langle A, \varrho \rangle := \mathrm{Tr}(A\varrho)$ is the mean value of observable $A$ in the state $\varrho$, $\{a, b, \ldots\}$ is the set of elements $a, b, \ldots$). The operator $L_1$ (and therefore $L$) is not an arbitrary linear operator in $B_1$, but such that there exists a dynamical semigroup $\Lambda(t)$, $t \geqslant 0$, acting in $B_1$ (*cf.* Definition 1 in Part I and conditions (2.25)–(2.29) contained in this definition) for which

$$L\varrho = \text{s-}\lim_{t \to +0} \frac{\Lambda(t)\varrho - \varrho}{t}, \ \varrho \in D(L) \Lambda(t)\varrho = e^{Lt}\varrho, \ t \geqslant 0, \ \varrho = \varrho(0). \tag{2.5}$$

Following Kossakowski (1972) we shall construct some examples of $\Lambda(t)$ and $L$ which will be useful in the following. We give only final results since proofs may be found in Kossakowski's paper (1972).

Let us consider first a linear harmonic oscillator with frequency $\omega$ and Hamiltonian $\omega a^* a$, where $a^*$ and $a$ are the creation and annihilation operators, respectively. We shall construct two examples of dissipation operators.

1) We write $\varrho(0) =: \varrho$ and define

$$\varrho(t) = \Lambda(t)\varrho := \int_{z\in C} U(t, z)\varrho U^*(t, z)p(t, z)d^2z, \quad t \geqslant 0, \tag{2.6}$$

where $C$ is the complex plane, $d^2z = dxdy$, $z = x+iy$, $x, y \in R$ = the real axis, and $(\bar{z} = x-iy)$

$$U(t, z) := \exp(-i\omega a^* at)\exp(za^* - \bar{z}a), \tag{2.7}$$

$$p(t, z) := \frac{1}{8\pi t\eta^2}\exp\left(-\frac{|z|^2}{4t\eta}\right), \quad \eta > 0. \tag{2.8}$$

We obtain

$$\frac{d\varrho(t)}{dt} = L\varrho(t) = \frac{1}{i}[\omega a^* a, \varrho(t)] + \eta[[a, \varrho(t)], a^*], \quad t \geqslant 0 \tag{2.9}$$

and hence

$$L_0 = \frac{1}{i}[\omega a^* a, .], \quad L_1 = \eta[[a, .], a^*]. \tag{2.10}$$

2) We put

$$\varrho(t) = \Lambda(t)\varrho := \int_{z\in C} V(t, z)\varrho V^*(t, z)p(t, z)d^2z, \quad t \geqslant 0 \tag{2.11}$$

where

$$V(t, z) = \exp\left\{-t(\kappa+i\omega)\left(a^* a - \frac{\bar{z}a}{1-\exp[-t(\kappa+i\omega)]}\right)\right\}, \quad \kappa > 0, \tag{2.12}$$

$$p(t, z) = \frac{1}{\pi(1-e^{-2\kappa t})}\exp\left(-\frac{|z|^2}{1-e^{-2\kappa t}}\right). \tag{2.13}$$

Now we obtain $(t \geqslant 0)$

$$\frac{d\varrho(t)}{dt} = L\varrho(t) = \frac{1}{i}[\omega a^* a, \varrho(t)] + \kappa([a, \varrho(t)a^*] + [a\varrho(t), a^*]), \tag{2.14}$$

and therefore

$$L_0 = \frac{1}{i}[\omega a^* a, .], \quad L_1 = \kappa([a, .\, a^*] + [a\, ., a^*]). \tag{2.15}$$

The dissipation operator $L_1$ occurring in (2.10) may be called the diffusion or Brownian motion or Langevin force operator because of the form of (2.8) and the character of its action on $\varrho(t)$ causing an increase of mean energy

$$E(\varrho(t)) = \langle H, \varrho(t)\rangle := \mathrm{Tr}(H\varrho(t)), H := \omega a^* a, \tag{2.16}$$

*i.e.*,

$$\frac{d}{dt} E(\varrho(t)) = \langle H, (L_0+L_1)\varrho(t)\rangle = \eta, \quad \eta > 0. \tag{2.17}$$

Independently of the initial state, the final state is always mixed, we have heie therefore the case 3 given in Table I of Part I.

On the other hand, in the example 2) the dissipation operator $L_1$ in (2.15) may be called the friction operator since it causes, in general, a decrease of both entropy and mean energy as we have

$$\underset{t\to\infty}{s\text{-lim}}\ \varrho(t) = P_0, \tag{2.18}$$

where $P_0$ is the projection operator on the vacuum-state-vector 0 (the ground state), independently of the initial state $\varrho$ (an ergodic motion). Thus

$$s(\varrho(\infty)) = E(\varrho(\infty)) = 0; \quad s(\varrho(t)) = -\mathrm{Tr}\,(\varrho(t)\ln \varrho(t)). \tag{2.19}$$

We have therefore case 2 of the classification given in Part I, Table I.

The dissipation operators $L_1$ in (2.10) and (2.15) were found by means of some plausibility considerations just for application in laser theory first by Weidlich and then frequently used in publications of the Stuttgart School (Weidlich, Risken and Haken 1967, 1967a, 1967b, Gnutzmann and Weidlich 1968, Weidlich, Risken, Haken and Gnutzmann 1969, Gnutzmann 1969). Actually, the combination of both the dissipation operators was used

$$\frac{d\varrho(t)}{dt} = \frac{1}{i}[\omega a^* a, \varrho(t)] + \kappa([a, \varrho(t)a^*] + [a\varrho(t), a^*]) + \\ + \eta[[a, \varrho(t)], a^*]. \tag{2.20}$$

It may be easıly shown that (2.20) is the most general expression containing only terms with one $a$ and one $a^*$, and preserving the trace of $\varrho(t)$. Kossakowski (1972a) proved that any sum of dissipators is a dissipator, so (2.20) presents a correct equation of motion in our sense.

Eq. (2.20) describes also an ergodic motion since, independently of the initial[1] state, we obtain asymptotically the stationary Gibbsian canonical final state

$$\underset{t\to\infty}{s\text{-lim}}\ \varrho(t) = Z^{-1}(\beta)\exp(-\beta a^* a), \tag{2.21}$$

$$Z(\beta) := \mathrm{Tr}\,(-\beta a^* a) = (1-e^{-\beta})^{-1}, \tag{2.22}$$

---

[1] Actually, only if the initial state is diagonal in $H$. $T_0$ obtain such a situation for a completely arbitrary initial state we have to add to the right-hand side of (2.20) the dissipator of "off-diagonal friction" found by Kossakowski (1972) and applied to this problem by Ingarden (1973).

where the (dimensionless) inverse temperature $\beta = T^{-1}$ ($T$ — temperature) is a function of $\kappa$ and $\eta$ only,

$$\beta = \frac{1}{T} = -\ln\frac{\eta}{2\kappa+\eta} \approx \frac{2\kappa}{\eta} \quad \text{(the latter for } \eta \gg \kappa\text{)}, \tag{2.23}$$

the right-hand side formula giving the "high temperature approximation". The rightness of (2.21)–(2.23) may be checked by inserting (2.21) into (2.20) with the time derivative equal to 0. We obtain

$$(2\kappa+\eta)ae^{-\beta a^*a}a^*e^{\beta a^*a}+\eta a^*e^{-\beta a^*a}ae^{\beta a^*a} =$$

$$= (2\kappa+\eta)a^*a+\eta, \tag{2.24}$$

which after using the well-known relations (*cf.* Haken 1970, p. 298, Eqs (X.2.11) and (X.2.12))

$$e^{-\beta a^*a}a^*e^{\beta a^*a} = a^*e^{-\beta}, \quad e^{-\beta a^*a}ae^{\beta a^*a} = ae^{\beta} \tag{2.25}$$

gives an identity only under condition (2.23). We used also the definition relation of $a$ and $a^*$

$$[a, a^*] = 1. \tag{2.26}$$

We now obtain

$$E(\varrho(\infty)) = \frac{\omega e^{-\beta}}{1-e^{-\beta}} = \frac{\omega\eta}{2\kappa}, \tag{2.27}$$

$$s(\varrho(\infty)) = -\ln(1-e^{-\beta})+\frac{\beta e^{-\beta}}{1-e^{-\beta}} = -\ln\left(1-\frac{\eta}{2\kappa+\eta}\right)+$$

$$+\frac{\eta}{2\kappa}\ln\frac{2\kappa+\eta}{\eta}. \tag{2.28}$$

We see that in general the mean energy and the entropy of the final state may be greater or smaller than (or equal to) the respective properties of the initial state (an exchange of energy and information with the environment). If, however, we consider only the macrostate fixed by the mean energy (2.27) (*i.e.*, the class of all states with the mean energy (2.27), *cf.* Part I), the motion (2.20) with the initial state in the macrostate does not change the mean energy while the entropy increases up to the equilibrium maximum value (2.28). On this simple example we see how our quantum irreversible thermodynamics works giving results similar to the classical well-known ones (which, however, as far as we know never have been formulated in such a precise way).

Now we go over to the second system of importance for our later investigations, namely a single spin $\frac{1}{2}$ system, the simplest quantum system possible, whose Hilbert space is two-dimensional. In this case it is possible to give the most general explicit theory of the evolution equation and this was done by Kossakowski (1972b). We give here also only the final results of his investigations. Any linear operator $X$ in 2-dimensional Hilbert space can be presented in the form

$$X = \tfrac{1}{2}(x_0\sigma_0 + \sum_{i=1}^{3} x_i\sigma_i) = \tfrac{1}{2}(x_0\sigma_0 + \boldsymbol{x}\boldsymbol{\sigma}), \tag{2.29}$$

where $x_0, \ldots, x_3 \in \boldsymbol{C}$ and

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.30}$$

We have the well-known relations

$$\sigma_s^2 = \sigma_0 \, (s = 0, 1, 2, 3), \qquad \sigma_j\sigma_k = \delta_{jk}\sigma_0 + i\sum_{l=1}^{3} \varepsilon_{jkl}\sigma_l$$

$$(j, k = 1, 2, 3), \tag{2.31}$$

where $\delta_{jk}$ and $\varepsilon_{jkl}$ are the Kronecker and Levi-Civita symbols, respectively. $X$ is hermitian iff $x_0, \ldots, x_3 \in \boldsymbol{R}$ and then we obtain from (2.29)

$$X = \tfrac{1}{2}\begin{pmatrix} x_0 + x_3, & x_1 - ix_2 \\ x_1 + ix_2, & x_0 - x_3 \end{pmatrix}. \tag{2.32}$$

Its eigenvalues are

$$\xi_{1,2} = \tfrac{1}{2}(x_0 \pm (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}) = \tfrac{1}{2}(x_0 \pm |\boldsymbol{x}|). \tag{2.33}$$

Therefore, $X$ is positive definite iff

$$x_0 \geqslant 0, \qquad |\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} \leqslant x_0. \tag{2.34}$$

Any density operator $\varrho \geqslant 0$, $\operatorname{Tr} \varrho = 1$ can be presented as

$$\varrho = \tfrac{1}{2}(\sigma_0 + \boldsymbol{x}\boldsymbol{\sigma}), \quad |\boldsymbol{x}| \leqslant 1. \tag{2.35}$$

We shall write in general

$$\varrho(x_0, \boldsymbol{x}) := \tfrac{1}{2}(x_0\sigma_0 + \boldsymbol{x}\boldsymbol{\sigma}) =: \tfrac{1}{2}(x_0\sigma_0 + \hat{x}), \tag{2.36}$$

so the density operators have the form $\varrho(1, \boldsymbol{x})$. It can be shown (Kossakowski 1972b) that the most general infinitesimal operator (superoperator) transforming a $\varrho(1, \boldsymbol{x})$ into a $\boldsymbol{x}(1, \boldsymbol{y})$ has the form

$$L\varrho = \frac{1}{i}[\hat{b}, \varrho] + \frac{1}{4}\sum_{j=1}^{3}\left\{\gamma_j - \frac{\gamma_1 + \gamma_2 + \gamma_3}{2}\right\}[\hat{e}_j, [\hat{e}_j, \varrho]], \tag{2.37}$$

where $\boldsymbol{b}$ is a 3-dimensional real vector and $\boldsymbol{e}_j$ $(j = 1, 2, 3)$ form an orthonormal vector base in $\boldsymbol{R}^3$, while

$$\gamma_1, \gamma_2, \gamma_3 \geqslant 0 \tag{2.38}$$

are dissipation constants of our problem. In particular, we may put (we shall use this case later on for our laser model)

$$e_j = (\delta_{ij})\ (j = 1, 2, 3), \quad \gamma_1 = \gamma_2 = \gamma_\perp > 0, \quad \gamma_3 = \gamma_{||} > 0, \tag{2.39}$$

and obtain

$$L_1\varrho = -\tfrac{1}{2}[\gamma_\perp(\sigma_1\varrho\sigma_1+\sigma_2\varrho\sigma_2)+\gamma_{||}\sigma_3\varrho\sigma_3]- -\tfrac{1}{2}(2\gamma_\perp+\gamma_{||})(\varrho-\sigma_0) = -\tfrac{1}{8}\{\gamma_{||}([\sigma_1,[\sigma_1,\varrho]]+[\sigma_2,[\sigma_2,\varrho]]+ +(2\gamma_\perp-\gamma_{||})[\sigma_3,[\sigma_3,\varrho]]\}. \tag{2.40}$$

### 3. *Pumping operator*

The laser action is frequently described as a balance between dissipation and pumping. This is a true picture, although distinction between the dissipation operators and the pumping ones cannot be carried out in a clear cut way. Indeed, in our examples above the dissipation operators (2.14) and (2.20) may be considered also as pumping operators since they "pump" any initial state into the final states (2.18) and (2.21), respectively. In these processes the entropy of the system can sometimes decrease, therefore the system is then ordered, "pumped". But, as we mentioned above, if we start our motion not from any state (any element of the set $W$, *cf.* Part I (2.2)), but from any element of the macrostate $M$ (*cf.* Part I (3.4)) defined by the mean energy (2.27), then pumping is excluded and we have only a dissipation (the entropy increases). Therefore, the concepts of dissipation and pumping are relative to the set of initial states considered. The same will be true with respect to the class of "pumping operators" which will be considered in this section, although now they should primary "pump" and not "dissipate" (and they do so with respect to all initial states with entropy greater than the entropy of the final state).

In this case we are interested (in view of the application to lasers) only in the spin $\frac{1}{2}$ system, *i.e.* a 2-dimensional Hilbert space. If (*cf.* (2.36))

$$\varrho(x_0, \boldsymbol{x}) = \tfrac{1}{2}(x_0\sigma_0+\boldsymbol{x}\boldsymbol{\sigma}) = \varrho^*(x_0, \boldsymbol{x}) \tag{3.1}$$

is a hermitian operator ($x_0, \boldsymbol{x} \in \boldsymbol{R}$), we have

$$\mathrm{Tr}\ \varrho(x_0, \boldsymbol{x}) = x_0, \quad ||\varrho(x_0, \boldsymbol{x})||_1 = \tfrac{1}{2}\{|x_0+\boldsymbol{x}|+|x_0-\boldsymbol{x}|\}, \tag{3.2}$$

*cf.* Part I (2.3). Iff $\varrho(x_0, \boldsymbol{x}) \geqslant 0$, we have (2.34) and

$$\mathrm{Tr}\ \varrho(x_0, \boldsymbol{x}) = ||\varrho(x_0, \boldsymbol{x})||_1. \tag{3.3}$$

For density operators $\varrho \in W(\varrho \geqslant 0, \mathrm{Tr}\ \varrho = 1)$ we have *cf.* (2.35),

$$\varrho = \varrho(1, \boldsymbol{x}), \quad |\boldsymbol{x}| \leqslant 1. \tag{3.4}$$

Following Kossakowski (1972b) we construct now a class of pumping operators by means of an inhomogeneous linear transformation (after that, however, we can "compress" this class into the class of homogeneous transformations by means of relation (2.24) of Part I, due to $\operatorname{Tr}\varrho = 1$). Let us take first a homogeneous linear transformation depending on time $t \geqslant 0$

$$A(t)\varrho(1, \boldsymbol{x}) = \varrho(1, A(t)\boldsymbol{x}), \tag{3.5}$$

where $A(t)\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$ is a homogeneous linear map such that

$$|A(t)\boldsymbol{x}| \leqslant |\boldsymbol{x}| \text{ for all } \boldsymbol{x} \in \boldsymbol{R}^3,\ A(t)A(s) = A(t+s),\ s, t \geqslant 0. \tag{3.6}$$

There exists a linear transformation $B$ in $\boldsymbol{R}^3$ such that

$$\frac{dA(t)}{dt} = A(t)B = BA(t),\ i.e.\ A(t) = e^{tB}. \tag{3.7}$$

Then we obtain

$$\frac{dA(t)\varrho(1, \boldsymbol{x})}{dt} = \varrho\left(0, \frac{dA(t)\boldsymbol{x}}{dt}\right) = \varrho(0, BA(t)\boldsymbol{x}), \tag{3.8}$$

or denoting

$$\varrho(t) = \varrho(1, \boldsymbol{x}(t)) := A(t)\varrho(1, \boldsymbol{x}),\ \boldsymbol{x}(t) := A(t)\boldsymbol{x},\ |\boldsymbol{x}| \leqslant 1,\ t \geqslant 0 \tag{3.9}$$

(of course, $\boldsymbol{x}(0) = \boldsymbol{x}$, $\varrho(0) = \varrho$)

$$\frac{d\varrho(t)}{dt} = \frac{d\varrho(1, \boldsymbol{x}(t))}{dt} = \varrho(0, B\boldsymbol{x}(t)) =: L\varrho(1, \boldsymbol{x}(t)) = L\varrho(t). \tag{3.10}$$

Let us take now a fixed density operator

$$\varrho_0 = \varrho(1, \boldsymbol{y}),\ |\boldsymbol{y}| \leqslant 1, \tag{3.11}$$

and define the inhomogeneous transformation connected with (3.5) ($\varrho = \varrho(1, \boldsymbol{x})$)

$$\tilde{A}(t)\varrho = A(t)\varrho + (\varrho_0 - A(t)\varrho_0) \tag{3.12}$$

or more explicitly

$$\tilde{A}(t)\varrho(1, \boldsymbol{x}) := \varrho(1, A(t)\boldsymbol{x} + \boldsymbol{y} - A(t)\boldsymbol{y}),\ |\boldsymbol{x}| \leqslant 1. \tag{3.13}$$

Let

$$Z(\boldsymbol{y}) := \sup_{t \geqslant 0} \sup_{|\boldsymbol{x}| \leqslant 1} |A(t)\boldsymbol{x} + \boldsymbol{y} - A(t)\boldsymbol{y}|, \tag{3.14}$$

$$F(L) := \{\boldsymbol{y} \in \boldsymbol{R}^3 : |\boldsymbol{y}| \leqslant 1,\ Z(\boldsymbol{y}) \leqslant 1\}. \tag{3.15}$$

We assume now that

$$\boldsymbol{y} \in F(L) \tag{3.16}$$

which is in general a stronger condition than (3.11) and which guarantees that

$$\varrho(t) = \varrho(1, \boldsymbol{x}(t)) := \tilde{A}(t)\varrho(1, \boldsymbol{x}) = \varrho(1, A(t)\boldsymbol{x} + \boldsymbol{y} - A(t)\boldsymbol{y}) \geqslant 0, \tag{3.17}$$

*i.e.* the preservation of positive definiteness of $\varrho(t)$ for $t > 0$ (while the preservation of trace, $\mathrm{Tr}\,\varrho(t) = 1$, is guaranteed by the form of (3.12) and (3.5)). We call $F(L)$ the **pumping domain** of the dissipation operator the dissipation operator $L$.

We obtain finally

$$\frac{d\varrho(t)}{dt} = L(\varrho(t)-\varrho_0), \quad \varrho_0 \in F(L), \quad t \geqslant 0. \tag{3.18}$$

In our case (2.40) we assume

$$\varrho_0 = \varrho_0(1, 0, 0, x), \quad |x| \leqslant f(\gamma_\perp, \gamma_{||}) \leqslant 1, \tag{3.19}$$

where $x$ may be called the **pumping parameter**. It can be shown (Kossakowski 1972b) that in this case $f(\gamma_\perp, \gamma_{||}) \equiv 1$, *i.e.* is independent of the dissipation constants, which considerably simplifies our calculations. Thus

$$|x| \leqslant 1. \tag{3.20}$$

## 4. *A laser model*

After the above preparation we are able to formulate a simple model of a solid state laser. Our model consists of two components:

1) a lasing substance composed of $N$ two-level atoms (magnetic ions) considered as unmoving in space and sufficiently distant one from the other that there is no direct dynamical interaction between themselves. The two energy levels, *e.g.* two Zeeman levels in a magnetic field, are different from the ground state level considered as distant in comparison to the difference of energy of the two levels (as the other, higher or lower levels of the atoms which are neglected in our model). The atoms are identical, but distinguishable through their positions in space which are different and numerated by the index $j = 1, 2, \ldots, N$. The atoms are described by "energetic" spins $\frac{1}{2}$ with index $j$, their 3rd component ($z$-component), $\frac{1}{2}\sigma_3^j$, *cf.* (2.30), being proportional to the energy operator of the $j$th atom, while the interaction with the electromagnetic fields is determined by the dipol moment operator which we take as proportional to 1st component ($x$-component) of the $j$th spin, *i.e.* $\frac{1}{2}\sigma_1^j$, *cf.* (2.30). Other levels of the atoms are considered as a part of the environment and all the total effect of the environment is described by the dissipation operator $L_1^j$ for each atom, *cf.* (2.40). The atoms are pumped by the pumping operator (3.19) with (3.18); (3.20).

2) an electromagnetic cavity with one mode in exact resonance with the energy difference of the lasing atoms, while the energy levels of other modes are considered as distant and as forming a part of the environment. The mode in question is described by a linear oscillator with frequency $\omega$, while the effect of environment is described by the dissipation operator as in (2.20).

According to our assumptions we may write down our equation of motion (dynamical semi-group, master equation) in the following form ($t \geqslant 0$)

$$\frac{d\varrho(t)}{dt} = \frac{1}{i}[H, \varrho(t)] + L_1^a(\varrho(t)-\varrho_0(x)) + L_1^e\varrho(t), \tag{4.1}$$

(where $\hbar = 1$, $g$ — the coupling constant, $a$ — atoms, $e$ — electromagnetic field)

$$H := \tfrac{1}{2}\omega \sum_{j=1}^{N} \sigma_3^j + \omega a^* a - g(a+a^*) \sum_{j=1}^{N} \sigma_1^j, \tag{4.2}$$

$$L_1^a \varrho = (L_1^1 \times L_1^2 \times \ldots \times L_1^N)\varrho, \tag{4.3}$$

where

$$L_1^j \bullet = -\tfrac{1}{2}[\gamma_\perp(\sigma_1^j \bullet \sigma_1^j + \sigma_2^j \bullet \sigma_2^j) + \gamma_{||}\sigma_3^j \bullet \sigma_3^j] - $$
$$- \tfrac{1}{2}(2\gamma_\perp + \gamma_{||})(\bullet - \sigma_0^j),$$

$$L_1^e \varrho := \kappa([a, \varrho a^*] + [a\varrho, a^*]) + \eta[[a, \varrho], a^*], \tag{4.4}$$

$$\varrho_0(x) = \frac{1}{2^N}(\sigma_0^1 + x\sigma_3^1) \times \ldots \times (\sigma_0^N + x\sigma_3^N), \qquad xN =: X, \tag{4.5}$$

$\varrho(t)$ and all other operators acting in the Hilbert space defined as the direct product of $N$ 2-dimensional Hilbert spaces of each atom and the infinite dimensional separable Hilbert space of the harmonic oscillator (the mode) (I is the identity operator in the latter space). All dissipation constants $\gamma_\perp$, $\gamma_{||}$, $\kappa, \eta$ are positive.

## 5. *A 6-temperature thermodynamics of a laser*

While in the preceding section we considered the (generalized) dynamics of a laser, now we go over to its thermodynamics in the sense and according to the method explained in Section 3 of Part I. We assume the following thermodynamically regular system of linearly independent observables of our laser (Ingarden 1971)

$$A_1 := \sum_{j=1}^{N} \sigma_1^j, \quad A_2 := \sum_{j=1}^{N} \sigma_2^j, \quad A_3 := \sum_{j=1}^{N} \sigma_3^j, \tag{5.1}$$

$$A_4 := (a^* + a), \; A_5 := i(a^* - a), \; A_6 := a^* a. \tag{5.2}$$

All the observables are real and dimensionless, and have clear physical meaning: $A_3$ and $A_6$ are proportional to the energy operators of the (free) $a$- and $e$-systems ($a$ — atoms, $e$ — electromagnetic field), respectively, $A_1$ and $A_2$ proportional to the two independent components of the dipol moment of the atomic system (in (4.2) we assumed that only one is essential which is possible by taking the corresponding axis in the direction of the electric field), while $A_4$ and $A_5$ are proportional to the electric and magnetic field, respectively. The system (5.1)–(5.2) is the smallest which can be used for describing the laser action in correspondence to the well-known results (*cf.* Risken 1968, Haken 1970). In the next section we shall consider (using some results of Kojro) a larger system of observables and shall see how this description is richer, but preserves all general features of our present description. In such a way we shall see that no subjective element exists in our method, except the choice of the number of details which is always free by an incomplete description.

For the moment $t = 0$ we assume the following ensemble mean values as given (not necessarily numerically)

$$\langle A_n, \varrho \rangle = \mathrm{Tr}\,(A_n \varrho) = U_n \; (n = 1, \ldots, 6). \tag{5.3}$$

Using the method presented in Part I Eq. (3.18) and using the Liouvillian (4.1) we obtain finally the following closed system of differential equations for $U_n(t)$ (the dot over $U_n$ denotes the time derivative, we drop $(t)$):

$$\dot{U}_1 = -\omega U_2 - \gamma_\perp U_1, \quad \dot{U}_2 = \omega U_1 - \gamma_\perp U_2 + 2gU_3U_4,$$

$$\dot{U}_3 = \gamma_{||}(X - U_3) - 2gU_2U_4, \tag{5.4}$$

$$\dot{U}_4 = \omega U_5 - \kappa U_4, \quad \dot{U}_5 = -\omega U_4 - \kappa U_5 + 2gU_1,$$

$$\dot{U}_6 = \eta - 2\kappa U_6 + gU_1U_5. \tag{5.5}$$

The automatic factorisation of the correlations between $a$- and $e$-systems (occurs in (5.4) and (5.5) in the terms with coupling constant $g$, except in the expression for $\dot{U}_5$) is caused by the fact that $\sigma_M(t)$, *cf.* (3.11) in Part I, maximizes entropy at any time $t \geqslant 0$ only by $U_n(t)$ which do not contain correlations. Therefore, at any time $t \geqslant 0$ (the macrostate $M$ is defined by (5.3))

$$\sigma_M(t) = \sigma_M^a(t) \times \sigma_M^e(t). \tag{5.6}$$

In order to exclude the rapid fluctuation with frequency $\omega$ we introduce first the rotating variables $\tilde{U}_n$

$$U_1 = \tilde{U}_1 \cos \omega t - \tilde{U}_2 \sin \omega t, \quad U_2 = \tilde{U}_1 \sin \omega t + \tilde{U}_2 \cos \omega t, \quad U_3 = \tilde{U}_3, \tag{5.7}$$

$$U_4 = \tilde{U}_4 \cos \omega t + \tilde{U}_5 \sin \omega t, \quad U_5 = -\tilde{U}_4 \sin \omega t + \tilde{U}_5 \cos \omega t, \quad U_6 = \tilde{U}_6. \tag{5.8}$$

We obtain

$$\dot{\tilde{U}}_1 + \gamma_\perp \tilde{U}_1 = g\tilde{U}_3(\tilde{U}_4 \sin 2\omega t + \tilde{U}_5(1 - \cos 2\omega t)), \tag{5.9}$$

$$\dot{\tilde{U}}_2 + \gamma_\perp \tilde{U}_2 = g\tilde{U}_3(\tilde{U}_4(1 + \cos 2\omega t) + \tilde{U}_5 \sin 2\omega t)), \tag{5.10}$$

$$\dot{\tilde{U}}_3 + \gamma_{||}(\tilde{U}_3 - X) = -g(\tilde{U}_1\tilde{U}_4 \sin 2\omega t + \tilde{U}_1\tilde{U}_5(1 - \cos 2\omega t) + \\ + \tilde{U}_2\tilde{U}_4(1 + \cos 2\omega t) + \tilde{U}_2\tilde{U}_5 \sin 2\omega t), \tag{5.11}$$

$$\dot{\tilde{U}}_4 + \kappa \tilde{U}_4 = g(-\tilde{U}_1 \sin 2\omega t + \tilde{U}_2(1 - \cos 2\omega t)), \tag{5.12}$$

$$\dot{\tilde{U}}_5 + \kappa \tilde{U}_5 = g(\tilde{U}_1(1 + \cos 2\omega t) - \tilde{U}_2 \sin 2\omega t), \tag{5.13}$$

$$\dot{\tilde{U}}_6 + 2\kappa \tilde{U}_6 = \eta + \tfrac{1}{2} g(\tilde{U}_1\tilde{U}_4 \sin 2\omega t + \tilde{U}_1\tilde{U}_5(1 + \cos 2\omega t) + \\ + \tilde{U}_2\tilde{U}_4(1 - \cos 2\omega t) + \tilde{U}_2\tilde{U}_5 \sin 2\omega t). \tag{5.14}$$

We apply now the well-known method of the rotating-wave approximation (*cf. e.g.* Bogolubov and Mitropolski 1965), we consider the Fourier extensions in $t$ of the right hand sides of (5.5)–(5.14) and take only the first, constant term. We get

$$\dot{\tilde{U}}_1 + \gamma_\perp \tilde{U}_1 = g\tilde{U}_3\tilde{U}_5, \tag{5.15}$$

$$\dot{\tilde{U}}_2+\gamma_\perp\tilde{U}_2 = g\tilde{U}_3\tilde{U}_4, \tag{5.16}$$

$$\dot{\tilde{U}}_3+\gamma_{||}(\tilde{U}_3-X) = -g(\tilde{U}_1\tilde{U}_5+\tilde{U}_2\tilde{U}_4), \tag{5.17}$$

$$\dot{\tilde{U}}_4+\kappa\tilde{U}_4 = g\tilde{U}_2, \tag{5.18}$$

$$\dot{\tilde{U}}_5+\kappa\tilde{U}_5 = g\tilde{U}_1, \tag{5.19}$$

$$\dot{\tilde{U}}_6+2\kappa\tilde{U}_6 = \eta+\tfrac{1}{2}g(\tilde{U}_1\tilde{U}_5+\tilde{U}_2\tilde{U}_4). \tag{5.20}$$

We are interested in the asymptotic stationary solutions of our equations (for $t = \infty$), and we may obtain them easily from the above Eqs by equating the time derivatives to 0. We obtain, however, two solutions of such an algebraic problem:

1) when the pumping $X \leqslant \frac{\kappa\gamma_\perp}{g^2} =: X_0 =$ the pumping threshold:

$$\tilde{U}_1 = \tilde{U}_2 = \tilde{U}_4 = \tilde{U}_5 = 0, \quad \tilde{U}_3 = X, \quad \tilde{U}_6 = \frac{\eta}{2\kappa}. \tag{5.21}$$

This is the ergodic solution (independent of the initial conditions) occurring in the usual thermodynamics (one temperature for each subsystem), for the *e*-system (linear oscillator) exactly the same which we obtained in Section 2, *cf.* (2.27), where $E = \omega U_6$ since $H$ in (2.16) is $\omega A_6$, *cf.* (5.2). We see that the both systems are decoupled and that the laser does not work;

2) when $X > X_0$:

$$\tilde{U}_4 = \frac{g}{\kappa}\tilde{U}_1, \quad \tilde{U}_5 = \frac{g}{\kappa}\tilde{U}_2, \quad \tilde{U}_3 = X_0, \quad \tilde{U}_6 = \frac{\eta}{2\kappa}+\frac{\gamma_{||}}{4\kappa}(X-X_0). \tag{5.22}$$

In this case not all $\tilde{U}_n$'s are fixed by the constants occurring in the thermodynamic equations of motion (5.15)–(5.20), two of them are free ($\tilde{U}_1$ and $\tilde{U}_2$, or $\tilde{U}_4$ and $\tilde{U}_5$), each pair being proportional to the other and being fixed only by the initial conditions. Therefore our solution is semi-ergodic, as we may call this situation: part of the final mean values are fixed by the equations only and are independent of the initial conditions, while the other depend on the initial conditions. The free mean values are just the quantities which may be called the signal parameters (*cf. e. g.* Louisell 1964, p. 245), the mean electric and magnetic fields, $\tilde{U}_4$ and $\tilde{U}_5$.

We see that below threshold we have only two independent mean values different from 0 (therefore, also two temperatures of the two subsystems), $\tilde{U}_3$ and $\tilde{U}_6$, above the threshold we have all the six mean values in general different from 0, which, however, are mutually coupled. Since we can easily observe only the *e*-system, of real importance are actually only $\tilde{U}_4$, $\tilde{U}_5$, $\tilde{U}_6$, but then we may calculate the other mean values from our formulae.

Our results are in exact agreement with the well-known results of the existing theories (Risken 1968, p. 269, Haken 1970) and correspond well to the experiments on lasers.

Now we may calculate explicitly the temperatures corresponding to the found mean values in the asymptotic solutions (*cf.* Ingarden 1969, 1971).

We obtain the following forms of the representative $\sigma_M$– states for our subsystems $a$ and $e$ in the asymptotic solution ($t = \infty$), *cf.* (5.6),

$$\sigma_M^a(\infty) = Z^{-1}(\beta_1, \beta_2, \beta_3) \exp(-\beta_1 A_1 - \beta_2 A_2 - \beta_3 A_3), \tag{5.23}$$

where

$$Z(\beta_1, \beta_2, \beta_3) = 2 \cosh (\beta_1^2 + \beta_2^2 + \beta_3^2)^{\frac{1}{2}}, \tag{5.24}$$

$$\beta_i = -\frac{\tilde{U}_i}{\tilde{U}_1^2 + \tilde{U}_2^2 + \tilde{U}_3^2} \ln \frac{N + (\tilde{U}_1^2 + \tilde{U}_2^2 + \tilde{U}_3^2)^{\frac{1}{2}}}{N - (\tilde{U}_1^2 + \tilde{U}_2^2 + \tilde{U}_3^2)^{\frac{1}{2}}} \quad (i = 1, 2, 3) \tag{5.25}$$

under the condition of regularity of the problem

$$\tilde{U}_1^2 + \tilde{U}_2^2 + \tilde{U}_3^2 \leqslant N. \tag{5.26}$$

Further,

$$\sigma_M^e(\infty) = Z^{-1}(\beta_4, \beta_5, \beta_6) \exp(-\beta_4 A_4 - \beta_5 A_5 - \beta_6 A_6), \tag{5.27}$$

where

$$Z(\beta_4, \beta_5, \beta_6) = \frac{\exp\left(\dfrac{\beta_4^2 + \beta_5^2}{\beta_6}\right)}{1 - \exp(-\beta_6)}, \tag{5.28}$$

$$\beta_4 = \tfrac{1}{2} \tilde{U}_4 \ln \frac{\tau^2}{2 + \tau^2}, \quad \beta_5 = \tfrac{1}{2} \tilde{U}_5 \ln \frac{\tau^2}{2 + \tau^2}, \quad \beta_6 = -\ln \frac{\tau^2}{2 + \tau^2} \tag{5.29}$$

under the condition of regularity

$$\tau^2 := 2\tilde{U}_6 - \tfrac{1}{2}(\tilde{U}_4^2 + \tilde{U}_5^2) \geqslant 0. \tag{5.30}$$

By the way we notice that knowing the generalized partition functions $Z$ (5.24) and (5.28) for both systems we can calculate any thermodynamic function for these systems (entropies, generalized specific heats, *etc.*).

Now we may calculate the inverse temperatures $\beta_n (n = 1, \ldots, 6)$ below and above the threshold:

1) below the threshold, $X \leqslant X_0$

$$\beta_1 = \beta_2 = \beta_4 = \beta_5 = 0, \quad \beta_3 = \frac{1}{X} \ln \frac{N - X}{N + X}, \quad \beta_6 = \ln \frac{2\kappa + \eta}{\eta}, \tag{5.31}$$

the latter in agreement with (2.23),

2) above the threshold, $X > X_0$
(we denote $\tilde{U}^2 := \tilde{U}_1^2 + \tilde{U}_2^2 + \tilde{U}_3^2 = \tilde{U}_1^2 + \tilde{U}_2^2 + X_0^2$)

$$\beta_1 = \frac{\tilde{U}_1}{\tilde{U}^2}\ln\frac{N-\tilde{U}}{N+\tilde{U}}, \quad \beta_2 = \frac{\tilde{U}_2}{\tilde{U}^2}\ln\frac{N-\tilde{U}}{N+\tilde{U}}, \quad \beta_3 = \frac{X_0}{\tilde{U}^2}\ln\frac{N-\tilde{U}}{N+\tilde{U}}, \tag{5.32}$$

$$\beta_4 = \frac{g}{2\kappa}\tilde{U}_2\ln\frac{\tau^2}{2+\tau^2}, \quad \beta_5 = \frac{g}{2\kappa}\tilde{U}_1\ln\frac{\tau^2}{2+\tau^2}, \quad \beta_6 = -\ln\frac{\tau^2}{2+\tau^2}, \tag{5.33}$$

where now

$$\tau^2 = \frac{\eta}{\kappa} + \frac{\gamma_{||}}{2\kappa}(X-X_0) - \frac{g^2}{2\kappa^2}(\tilde{U}_1^2+\tilde{U}_1^2). \tag{5.34}$$

If there is no signal ($\tilde{U}_1 = \tilde{U}_2 = 0$) we obtain above the threshold

$$\beta_1 = \beta_2 = \beta_4 = \beta_5 = 0, \quad \beta_3 = \frac{1}{X_0}\ln\frac{N-X_0}{N+X_0},$$

$$\beta_6 = \ln\left(1+\frac{4\kappa}{2\eta+\gamma_{||}(X-X_0)}\right). \tag{5.35}$$

For very strong pumping (supposing that it is not contradictory with the condition (3.20)), namely,

$$X \gg X_0 = \frac{\kappa\gamma_\perp}{g^2}, \quad X \gg \frac{2\eta}{\gamma_{||}}, \tag{5.36}$$

we obtain

$$\beta_6 = \frac{1}{T_6} \approx \frac{4\kappa}{\gamma_{||}X} \text{ or } T_6 \approx \frac{\gamma_{||}X}{4\kappa}, \tag{5.37}$$

*i. e.* the energetic temperature of the electromagnetic field is proportional to the pumping, while the energetic temperature of the atoms is constant and independent of the pumping (fixed by the threshold pumping $X_0$), *cf.* (5.35).

The threshold behaviour is typical to lasers, but also to superconductors. In fact, there is a close formal analogy between lasers and superconductors, in which the threshold pumping corresponds to the energy gap of the superconductor. In this analogy to the signal temperatures there correspond the Bogolubov potentials of the supercurrent. We see, therefore, that the macroscopic quantum behaviour of bulk matter, observed in lasers and superconductors, can be macroscopically described only by multitemperature thermodynamics (when we have only one temperature, below the threshold in lasers and above the energy gap in superconductors, these devices do not work). In such a way we have shown by means of these examples that multitemperature thermodynamics is essential for quantum macroscopic phenomena. Of course, in the special case when is no signal, or no supercurrent are present, the multitemperature thermodynamics degenerates into one--temperature thermodynamics, as was shown above although in this case we have actually two subsystems with two different temperatures (in lasers they are the *a*- and *e*-subsystems, in superconductors they are: the crystal lattice system and the free electron system). For technical applications, however, only the signal and supercurrent are essential, and then we have the full multitemperature thermodynamics.

### 6. *11-temperature thermodynamics of a laser*

The present author proposed to Z. Kojro in Toruń as a theme of his Ph. D. thesis, an investigation of higher than 6-temperature thermodynamics of our laser model. The thesis is not yet completed, but some of his partial results are interesting and important enough to discuss them here shortly in connection with our above study.

We mention first of all that the number of temperature chosen, although arbitrary in principle is practically limited by the convenience of calculation. Especially, it is useful to choose such systems of observables for each subsystem ($a$ and $e$ in our case) which form (for the coupling constant $g = 0$) Lie algebras of observables. We did so in Section 5, and the same was done by Kojro who chose the next Lie algebras in the $e$-system which contains the operator $A_6^2 = (a^*a)^2 = a^*aa^*a$ (*cf.* Ingarden 1971). The reason was to investigate the problem of the second order energetic temperature which seems to be observed in lasers (*cf.* Risken 1968, p. 279, Ingarden 1971, Weidlich *et al.* 1969).

Kojro took the following additional observables besides of ours (5.1)–(5.2)

$$A_7 := a^*a^*+aa,\ A_8 := i(a^*a-aa), \tag{6.1}$$

$$A_9 = a^*aa^*+a^*a^*a+aa^*a+a^*aa, \tag{6.2}$$

$$A_{10} = i(a^*aa^*+a^*a^*a-aa^*a-a^*aa), \tag{6.3}$$

$$A_{11} = a^*aa^*a. \tag{6.4}$$

Not repeating the investigations of Kojro which follow exactly along the same lines as ours, we give only his final results. All results for $n = 1, \ldots, 6$ are the same as ours, the new are only

1) below the threshold, $X \leqslant X_0$

$$\tilde{U}_7 = \tilde{U}_8 = \tilde{U}_9 = \tilde{U}_{10} = 0, \qquad \tilde{U}_{11} = \frac{\eta(\eta+\kappa)}{2\kappa^2} = \tilde{U}_6(2\tilde{U}_6+1), \tag{6.5}$$

2) above the threshold, $X > X_0$

$$\tilde{U}_7 = \frac{g^2}{2\kappa^2}(\tilde{U}_2^2-\tilde{U}_1^2), \qquad \tilde{U}_8 = \frac{g^2}{\kappa^2}\tilde{U}_1\tilde{U}_2, \tag{6.6}$$

$$\tilde{U}_9 = \frac{g}{3\kappa^2}\tilde{U}_2(3(2\eta+\kappa)+\gamma_{||}(X-X_0)+\frac{g^2}{\kappa}\tilde{U}_1\tilde{U}_2+$$

$$+\frac{g^2}{2\kappa}(\tilde{U}_2^2-\tilde{U}_1^2)), \tag{6.7}$$

$$\tilde{U}_{10} = \frac{g}{3\kappa^2}\tilde{U}_1(3(2\eta+\kappa)+\gamma_{||}(X-X_0)+\frac{g^2}{\kappa}\tilde{U}_1\tilde{U}_2+$$

$$+\frac{g^2}{2\kappa}(\tilde{U}_1^2-\tilde{U}_2^2)), \tag{6.8}$$

$$\tilde{U}_{11} = \frac{1}{4\kappa}\left(\eta+\frac{2\eta+\kappa}{\kappa}\left(\eta+\frac{\gamma_{||}}{2}(X-X_0)\right)+\frac{g^2}{2\kappa}(\tilde{U}_1^2+\right.$$

$$\left.+\tilde{U}_2^2)\left(2\eta+\kappa+\gamma_{||}(X-X_0)+\frac{g^2}{12\kappa^3}(\tilde{U}_1^2+\tilde{U}_2^2)^2\right)\right). \tag{6.9}$$

First of all we may state that below the threshold no new situation occurs. Indeed, $\sigma_M(\infty)$ is the same as before (5.31), (2.21)–(2.24), (5.23)–(5.24). For $\sigma_M^a(\infty)$ it is obvious since no new conditions are added, we shall also check that $\sigma_M^e(\infty)$ is the same. Indeed, we have with

$$\sigma_M^e(\infty) = (1-e^{-\beta})\exp(-\beta a^* a), \quad \beta = \ln\frac{2\kappa+\eta}{\eta}, \tag{6.10}$$

*cf.* (2.21), (2.22), (2.23),

$$\frac{\tilde{U}_6}{1-e^{-\beta}} = \mathrm{Tr}\,(a^* a e^{-\beta a^* a}) = \frac{e^{-\beta}}{(1-e^{-\beta})^2}, \tag{6.11}$$

$$\tilde{U}_{11} = (1-e^{-\beta})\,\mathrm{Tr}\,(a^* a a^* a e^{-\beta a^* a}) = -(1-e^{-\beta})\frac{d}{d\beta}\frac{\tilde{U}_6}{1-e^{-\beta}} =$$

$$= e^{-\beta}\frac{1+e^{-\beta}}{(1-e^{-\beta})^2} = \frac{\eta(\eta+\kappa)}{2\kappa^2}, \tag{6.12}$$

where we used (2.23). We see $\tilde{U}_{11}$ is the same as in (6.5), *i. e.* we have actually the same asymptotic solution as in Section 5 below the threshold.

Above the threshold the situation seems to be similar since we see that the formulae for $\tilde{U}_1-\tilde{U}_6$ are the same, while the values of $\tilde{U}_7-\tilde{U}_{11}$ are determined only by the constants of equations and $\tilde{U}_1$, $\tilde{U}_2$, as before, *i. e.* the same as in Section 5. Actually, however, the situation is different. We may check this for the simplest case when the signal vanishes ($\tilde{U}_1 = \tilde{U}_2 = 0$). Eqs (6.6)–(6.9) give then

$$\tilde{U}_7 = \tilde{U}_8 = \tilde{U}_9 = \tilde{U}_{10} = 0, \quad \tilde{U}_{11} = \frac{1}{4\kappa}\left(\eta+\frac{2\eta+\kappa}{\kappa}(\eta+\tfrac{1}{2}\gamma_{||}(X-X_0))\right). \tag{6.13}$$

On the other hand, if we use $\sigma_M^e(\infty)$ as in (6.10) but with $\beta = \beta_6$ as in (5.35) we obtain from (6.12) with $\beta = \beta_6$

$$\tilde{U}_{11} = \frac{1}{8\kappa^2}(2\eta+\gamma_{||}(X-X_0))(2\kappa+2\eta+\gamma_{||}(X-X_0)) = \tilde{U}_6(2\tilde{U}_6+1) \tag{6.14}$$

whis is essentially different from (6.13). We see therefore that above the threshold the problem with 11 mean values (5.1), (5.2), (6.1)–(6.4) gives in general different assymptotic solution $\sigma_M(\infty)$ and different inverse temperatures $\beta_m(m = 1, \ldots, 11)$ than the problem with only 6 mean values (5.1), (5.2). In the former case we obtain, in particular, a second order energetic but dimensionless temperature $\beta_{11}$, *i. e.* we have (for the simplest case when $\tilde{U}_1 = \tilde{U}_2 = 0$)

$$\sigma_M^e(\infty) = Z^{-1}(\beta_6, \beta_{11}) \exp(-\beta_6 a^* a - \beta_{11} a^* a a^* a). \tag{6.15}$$

Unfortunately, all temperatures of higher orders in $a$ and $a^*$ than those in (5.29) cannot be calculated exactly in a closed form and only approximative solutions (*e. g.* in high temperature approximation) can be found. Therefore, we resign here from any further calculations which will be given in the mentioned Ph. D. thesis of Mr Kojro. Anyway, however, we can state that our theory confirms positively the earlier theoretical and experimental findings of the second order energetic temperature in lasers above the threshold as mentioned above (Risken 1968, p. 279, where our terminology is, of course, not used, but the formula similar to (6.15) is given, Risken's formula (4.6)).

It seems, therefore, that our theory gives all essential effects known for lasers of this simple type. However, this theory may be developed much further, and more complicated types of lasers may be investigated by this method.

We should like to add yet a short general comment concerning the method. We see that adding new higher order observables (6.1)–(6.4) did not change essentially the result in this respect that (above the threshold of pumping) all mean values depend only on the constants contained in the equations of motion and on two signal parameters $\tilde{U}_1$ and $\tilde{U}_2$ (or $\tilde{U}_4$ and $\tilde{U}_5$). It is obvious that the same will occur after adding yet higher order observables. This fact is due to the semi-ergodic character of the motion. Below the threshold (as in usual thermodynamics of one temperature, *e. g.* of equation (2.20)), where the motion is ergodic, no information at all about mean values can influence the asymptotic solution. This is just the reason why thermodynamics is at all possible without performing measurements, except very few. The latter are necessary for measuring the dissipation and pumping constants, as well as signal parameters in the case of lasers (or supercurrent potentials in the case of superconductors, *etc.*). These constants may be measured either directly or by means of mean values of some observables, as (2.27), (5.21), (5.22), *etc.* We see that no subjective element is introduced into physics by information theory (as by probability theory in general). The only (seeming) subjectivity lies in which question is asked, but this is always so. The answers however are independent of us, they are objective, and depend (if they are true) on the nature of things. This statement is important since there is considerable misunderstanding among physicists and philosophers in this point, even Jaynes (1957) seems to belong to them since seemingly advocates the subjectivistic standpoint.

In contradistinction to the asymptotic solution (for $t = \infty$), the solutions for $0 \leqslant t < \infty$ (even in the ergodic and semiergodic cases) depend on initial conditions. It may be shown that in the rotating-wave approximation the problem of time evolution can be reduced to equations of the van der Pol or Rayleigh type (*cf.* Risken 1968, p. 270 and Anhang 1) and can be solved by means of the Bogolubov-Mitropolski (1962) solution (*cf.* Risken 1968, p. 270, Eq. (2.43)).

### 7. *Entropy fluctuations of a two-level atom (spin $\frac{1}{2}$)*

In our paper we have used the quasi-isoentropic approximation defined and generally discussed in Section 3 of Part I. As we have seen above this method gives reasonable and well corresponding to experiment results for lasers (if not mention the usual one-temperature thermodynamics). But it is interesting to know what is neglected in this method, anyhow in the case of a system simple enough that exact calculation is possible. The only such system is the spin $\frac{1}{2}$ or two-level atom system. Taking the Hamiltonian as in (4.2), but for one atom only, we obtain

$$H = \tfrac{1}{2}\omega\sigma_3, \tag{7.1}$$

for the sake of simplicity we assume no dissipation in our system (all dissipation and pumping constants vanish). In this case 3 linearly independent mean values determine the state uniquely (complete measurement), therefore we can take only two observables at most in our method of information thermodynamics, *e. g.*

$$A_1 = \sigma_1, A_3 = \sigma_3, \tag{7.2}$$

*cf.* (5.1), and put for $t = 0$

$$\langle A_1, \varrho\rangle = \mathrm{Tr}\,(\sigma_1\varrho) = U_1, \langle A_3, \varrho\rangle = \mathrm{Tr}(\sigma_3\varrho) = U_3. \tag{7.3}$$

In this case it will be simpler to use the Heisenberg picture since the equations of motion for $\sigma_1(t)$ and $\sigma_2(t)$ in this picture can be integrated immediately. The equations of motion have the form

$$\dot{\sigma}_1(t) = -\omega\sigma_2(t), \dot{\sigma}_2(t) = \omega\sigma_1(t), \tag{7.4}$$

and their solution are, of course, as follows

$$\sigma_1(t) = \sigma_1 \cos \omega t - \sigma_2 \sin \omega t,$$

$$\sigma_2(t) = \sigma_1 \sin \omega t + \sigma_2 \cos \omega t, \tag{7.5}$$

where $\sigma_1 = \sigma_1(0)$ and $\sigma_2 = \sigma_2(0)$. $\sigma_3(t) = \sigma_3(0) = \sigma_3$ is, of course, a constant of motion because of the Hamiltonian (7.1).

Now according to the method presented in Section 3 of Part I we calculate $\varrho_M$ (which in the Heisenberg picture is independent of time)

$$\varrho_M = Z^{-1} \exp(-\beta_1\sigma_1 - \beta_3\sigma_3) = Z^{-1} \exp(-\beta \boldsymbol{e\sigma}) =$$

$$= Z^{-1}(\sigma_0, \cosh \beta - \boldsymbol{e\sigma} \sinh \beta), \tag{7.6}$$

where we used the vectors

$$e := \left(\frac{\beta_1}{\beta}, 0, \frac{\beta_3}{\beta}\right), \quad (\beta := (\beta_1^2+\beta_2^2)^{\frac{1}{2}}),$$

$$\sigma := (\sigma_1, \sigma_2, \sigma_3). \tag{7.7}$$

Then

$$U_1(t) = \mathrm{Tr}\,(\varrho_M\sigma_1(t)) = \cos\omega t\,\mathrm{Tr}\,(\varrho_M\sigma_1) - \sin\omega t\,\mathrm{Tr}\,(\varrho_M\sigma_2) = \cos\omega t U_1, \tag{7.8}$$

$$U_3(t) = U_3, \tag{7.9}$$

where we used the relation (for $t = 0$)

$$\mathrm{Tr}(\varrho_M\sigma_2) = 0. \tag{7.10}$$

Calculation of $\sigma_M(t)$ gives finally

$$\sigma_M(t) = \tfrac{1}{2}\left\{\sigma_0 + \frac{2(\cos(\omega t)U_1\sigma_1 + U_3\sigma_3)}{1+\cos^2(\omega t)U_1^2+U_3^2}\right\}, \tag{7.11}$$

and we see that it is a periodical function with frequency $\omega$. The same frequency occurs in the exact thermodynamical entropy, *cf.* (3.19) of Part I,

$$S(t) := -\mathrm{Tr}\,(\sigma_M(t)\ln\sigma_M(t)). \tag{7.12}$$

The latter can be exactly calculated as follows (Kossakowski, a private communication)

$$S(t) = -\lambda_1(t)\ln\lambda_1(t) - \lambda_2(t)\ln\lambda_2(t), \tag{7.13}$$

where

$$\lambda_{1,2}(t) := \tfrac{1}{2}(1 \pm \lambda(t)), \quad \lambda(t) := \frac{2(\cos^2(\omega t)U_1^2+U_3^2)^{\frac{1}{2}}}{1+\cos^2(\omega t)U_1^2+U_3^2}. \tag{7.14}$$

A similar calculation was performed by Urbanik (1964), but with his earlier definition of macrostate and entropy in time $t$. One may see that our calculation and the final result are simpler, although both results are qualitatively similar.

Finally, we may say that what we neglect in our "quasi-isoentropic" method are very quick fluctuations of entropy and $\sigma_M(t)$ with optical frequency $\omega$. But in the above calculations of Sections 5 and 6 we also neglected phenomena with this frequency using the rotating-wave approximation, since they cannot be followed by macroscopic observation. Therefore, all of our method seems to be mathematically consistent and physically reasonable. We hope that its applications may be also practically fruitful.

The author thanks cordially Dr A. Kossakowski for his very useful and deep discussions, as well as for constant technical help in performing and checking our calculations. Thanks are also due to Mr Z. Kojro and Mr A. Jamiołkowski for their critical discussions and checking of some calculations.

## REFERENCES

Bogolubov, N. N., Mitropolski, I. A., *Les méthodes asymptotiques en théorie des oscillations non linéaires*, Gauthier-Villars, Paris 1962.
Fain, V. M., Khanin, Ya, F., *Quantum Electronics*, vol. 1, 2, Pergamon Press, Oxford 1967.
Fleck, J. A., *Phys. Rev.*, **149**, 301, 322 (1966).
Fleck, J. A., *Phys. Rev.*, **152**, 278 (1966a).
Gnutzmann, U., *Z. Phys.*, **222**, 283 (1969).
Gnutzmann, U., Weidlich, W., *Phys. Letters*, **27A**, 179 (1968).
Haken, H., *Laser Theory*, Hdb. Phys., vol. 25/2c, Springer, Berlin 1970.
Ingarden, R. S., *Acta Phys. Polon.*, **36**, 855 (1969).
Ingarden, R. S., *Bull. Acad. Polon. Sci. Sér. Math.*, **19**, 77 (1971).
Ingarden, R. S., *Postępy Fizyki*, to be published (1973) (in Polish).
Jaynes, E. T., *Phys. Rev.*, **106**, 620 (1957).
Kossakowski, A., *Rep. Math., Phys.*, **3**, 247 (1972).
Kossakowski, A., *Bull, Acad. Polon. Sci. Sér. Math.*, **20**, 1028 (1972a).
Kossakowski, A. *Bull. Acad. Polon. Sci. Sér. Math.*, to be published (1972b).
Louisell, W. H., *Radiation and Noise Quantum Electronics*, McGraw-Hill, New York 1964.
Risken, H., *Fortschr. Phys.*, **16**, 261 (1968).
Urbanik, K., *Trans. 3rd Prague Conf. Inform. Theory*, Czechosl. Acad. Sci., Prague 1964, p. 743.
Weidlich, W., Risken, H., Haken, H., *Z. Phys.*, **201**, 396 (1967).
Weidlich, W., Risken, H., Haken, H., *Z. Phys.*, **204**, 223 (1967a).
Weidlich, W., Risken, H., Haken, H., *Z. Phys.*, **206**, 335 (1967b).