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THE LOWEST ORDER POLARIZATION DIAGRAMS

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The lowest orders of the perturbation series for calculation of the effect of the electron-electron interaction on the polarization function are investigated.

The diagrams up to the third order in the electron-electron interaction are enumerated and expressions for calculation of the diagrams up to the second order are written down in the form in which the summation on the frequency variable has been performed.

1. *Introduction*

In various properties of solids, in particular of metals, the polarizability of the conduction electrons plays an important role and the effect of the electron-electron interaction on the polarizability is expected to be large. The effects of polarizability are usually studied by the dielectric formulation of the many-body problem, in particular by the investigation of the linear response function [1, 2] which is related to various observable properties of the system. In the calculation of the response function the electron-electron interaction can be taken into account by the perturbation theory with its diagrammatic techniques [1–12]. The Coulomb interaction between the electrons is usually described as a screened interaction [2, 4, 6, 11–13], the screening following from a summation of an infinite series of proper polarization diagrams. Also the phonon-induced interaction between the electrons is effectively a short-range interaction [14–17]. With the effective short-range interaction between the electrons and for the electrons in a plasma, between the electrons and ions, the response function can be evaluated in the lowest orders of perturbation theory. However liable to criticism, the calculations in the lowest orders of perturbation are of considerable value: if the sum of all perturbation orders does indeed

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give the correct response function, then there may be some other expansions of this function which could be discovered by studying the terms of the perturbations series [1, 18]. Hence the heuristic value of the study of the lowest order diagrams for the response function. Practically also, there is at present no much progress beyond the calculation of the very first terms of the perturbation series [2, 4–22].

We will here study in a systematic way the first three lowest orders of the perturbation series for the linear response function even though it is realized that the polarization diagrams with the electron-electron interaction included cannot be evaluated in analytical form.

2. Polarization diagrams

Properties of the electron system which are expressed in terms of an expectation value of a two-body operator are generally described by the polarization operator. The correlation energy of the electron gas is related to polarizability and can be expressed in terms of the polarization operator [1–4, 8–11].

The polarizability of the electron gas in a solid is expressed in terms of a frequency and wave-vector dependent dielectric function [8–12]. The conductivity and absorption coefficients are measured in optical investigations [5–7, 13, 14, 17–22]. The characteristic energy loss experiments on fast particles are related to the response function [4]. Thus the understanding of the behaviour of the electron gas requires an investigation of the response function. We want here to systematically enumerate the polarization diagrams in the lowest orders of the perturbation series, up to the third. We want to write down the integrations which have to be performed in the lowest orders up to the second.

TABLE I

Number of Green's function graphs[1]

Order	Linked		Uniform		Characteristic	
	s.	p.	s.	p.	s.	p.
0	1	1	1	1	1	1
I	2	6	1	4	2	1
II	10	50	4	27	2	7
III	74	518	27	248	10	63

[1] The first column in each entry gives the number of the single-particle Green's function graphs (s.), the second of the polarization Green's function graphs (p.) in the given order.

The response function is calculated as the sum of all topologically distinct linked graphs. The enumeration of graphs corresponds to the use of the Wick's theorem. It therefore takes in the given order the exchange effects into account [2, 4, 8, 10–12, 18, 23, 24]. In Table I we count up the graphs of the lowest orders from zero to three for the single-particle Green's function (see Figs 1, 2), and for the polarization Green's function

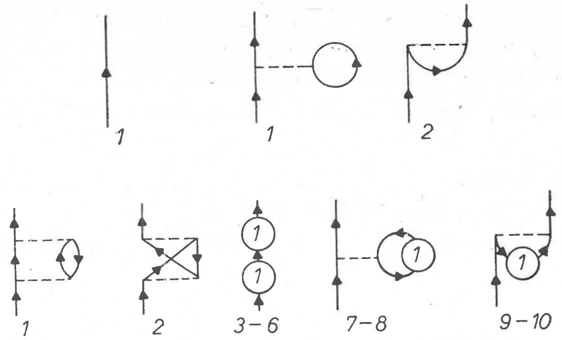

Fig. 1. The zeroth, first and second order graphs for the single-particle Green's function. Graphs *3* to *10* are pictured schematically, with ① meaning any first order graph of the single-particle Green's function

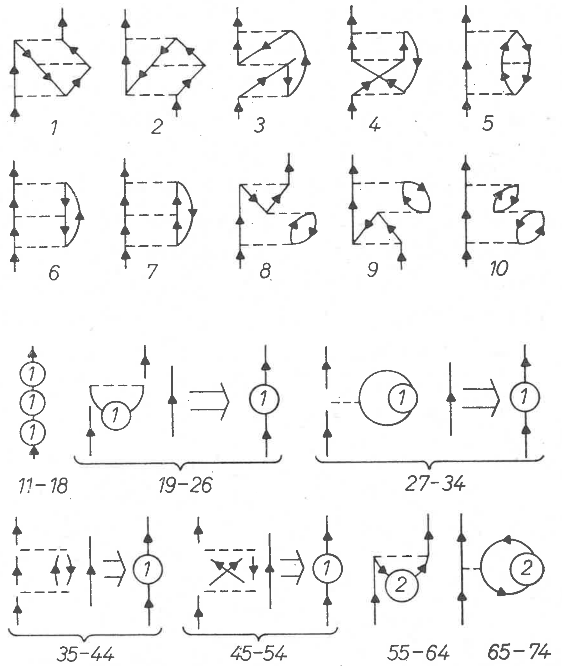

Fig. 2. The third order graphs for the single-particle Green's function. Graphs *1* to *10* are characteristic graphs. Graphs *11* to *74* are pictured schematically with ① or ② meaning any first or second order graphs respectively of the single-particle Green's function

(see Figs 3–10). In the Figures the Fermion propagators are represented by directed solid lines, the electron-electron effective interaction by a dashed horizontal line [1, 24]. The graphs describing the polarization operator have two vertices corresponding to correlation of two current densities. The first vertex in the highest point of each diagram, the second is the lowest.

For an electron gas in the uniform background of positive charge the contribution of diagrams having Fermion loops connected to the rest of a linked diagram by only a single interaction line, vanishes [1, 3]. Thus we omit such diagrams and remain only with the diagrams which we term "uniform" for short.

Diagrams which appear in a given order and

1) are not obtained by modyfying the Fermion propagators in the diagrams of lower orders,

2) cannot be divided into separate parts by cutting a single interaction line,

we term "characteristic" (see Figs 3, 4, 6, 7, 8, 9).

They give novel structures not describable in terms of lower order diagrams [24].

3. Contribution of the second order diagrams

The explicit expressions for contributions of diagrams of the orders zero, one, and two in the effective interaction will now be given in the form in which the summation on the frequency variable has been performed. We use the diagram technique of Luttinger and Ward [25], thus we give expressions valid for finite temperatures. We keep the terms which result from the repeating identical Fermion lines, *i. e.* which contain derivatives of the Fermi distribution function and go over into the delta function at zero temperature limit, the so-called anomalous contributions [25]. We write only expressions for the "uniform" diagrams which survive in a uniform background of positive charge.

We are in particular interested in the expressions for the conductivity tensor because it has clearly exhibited the tensorial symmetry properties [5, 6, 17, 20].

The conductivity tensor for the frequency ω and the wave vector $\boldsymbol{k}$ of the electromagnetic wave is

$$\sigma_{\mu\nu}(\boldsymbol{k}, \omega) = \frac{ie^2 n}{m\omega}\,\delta_{\mu\nu} - \frac{i}{\omega}\,M^+_{\mu\nu}(\boldsymbol{k}, \omega)$$

where n is the electron density and $M^+_{\mu\nu}(\boldsymbol{k}, \omega)$ is the analytic continuation from the infinite set of points $k_n = 2\pi\, in/\beta$ $(n > 0)$ to the real axis ω of the

$$M_{\mu\nu}(\boldsymbol{k}, k_n) = \left(\frac{e\hbar}{m}\right)^2 \frac{1}{v} \sum_{p,p'} p_\mu p'_\nu \langle S(\beta)\rangle_0^{-1} \int_0^\beta du\, e^{uk_n} \times$$

$$\times \langle T\{a^+_{p+\frac{k}{2}}(u)\, a_{p-\frac{k}{2}}(u)\, a^+_{p'-\frac{k}{2}}(0)\, a_{p'+\frac{k}{2}}(0)\, S(\beta)\}\rangle_0$$

$\langle\ldots\rangle_0$ is the thermal average for noninteracting particles at temperature $1/k_B\beta$, where k_B is the Boltzman constant.

$$S(\beta) = \exp\left[-\int_0^\beta du\, \frac{1}{2v} \sum_q U_q(u) \sum_{p,p'} a^+_{p+\frac{q}{2}}\, a^+_{p'-\frac{q}{2}}\, a_{p'+\frac{q}{2}}\, a_{p-\frac{q}{2}}\right]$$

and

$$U(\boldsymbol{q}, q_m) = \int_0^\beta du\, e^{uq_m} U_q(u)$$

with

$$q_m = 2\pi i m/\beta, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots,$$

is the Fourier transform of the interaction potential with the symmetry properties

$$U(\boldsymbol{q}, q_m) = U(\boldsymbol{q}, -q_m) = U(-\boldsymbol{q}, q_m).$$

The zeroth order polarization operator is

$$Q(\boldsymbol{q}, q_m) = \int \frac{d^3p}{(2\pi)^3} \frac{n_{p+\frac{q}{2}} - n_{p-\frac{q}{2}}}{\varepsilon_{p+\frac{q}{2}} - \varepsilon_{p-\frac{q}{2}} - q_m}.$$

The single particle energies are

$$\varepsilon_p = \frac{h^2 p^2}{2m}.$$

and the occupation numbers with the chemical potential μ are

$$n_p = \frac{1}{\exp \beta(\varepsilon_p - \mu) + 1}.$$

To write down the contributions of successive diagrams a succint notation is necessary since the expressions are very lengthy. Thus we introduce a shorthand notation which stresses the covariant character of the expressions.

We define for wave vectors $\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{q}$

1. The one symbol brace:

$$(p+k+q) = \varepsilon_{p+k+q}.$$

2. The two symbol braces:

$$(p+k+q, p+k+q') = \frac{n_{p+k+q} - n_{p+k+q'}}{\varepsilon_{p+k+q} - \varepsilon_{p+k+q'} - q_m + q'_{m'}},$$

$$(p+k+q, p+k+q')' = \frac{\partial n_{p+k+q}/\partial \varepsilon_{p+k+q}}{\varepsilon_{p+k+q} - \varepsilon_{p+k+q'} - q_m + q'_{m'}},$$

$$(p+k+q, p+k+q')'' = \frac{\partial^2 n_{p+k+q}/\partial \varepsilon^2_{p+k+q}}{\varepsilon_{p+k+q} - \varepsilon_{p+k+q'} - q_m + q'_{m'}},$$

$$(p+k+q, p+k+q')_l = \frac{n_{p+k+q} - n_{p+k+q'}}{[\varepsilon_{p+k+q} - \varepsilon_{p+k+q'} - q_m + q'_{m'}]^l}.$$

For $k = 0$ we need moreover

3. The three symbol brace:

$$(p+q, p, k_n) = \frac{n_{p+q} - n_p}{\varepsilon_{p+q} - \varepsilon_p + k_n - q_m}.$$

For convenience we will write down expressions for the current wave-vector $2\boldsymbol{k}$ instead of $\boldsymbol{k}$.

In the calculation of the response function to the long wave perturbing fields the limit of vanishing wave vector $\boldsymbol{k} = 0$ has to be considered. In this limit contributions of particular diagrams vanish or cancel when grouped together *e. g.* in pairs [55–7, 9–11, 17, 18, 20, 21]. Therefore we give after expression of a diagram its limit at $\boldsymbol{k} = 0$.

Numbers before each formula refer to numbers in the Figures, which display diagrams in given order of the perturbation.

Formulae for the contribution of the zeroth order diagram Fig. 3, No 0 are

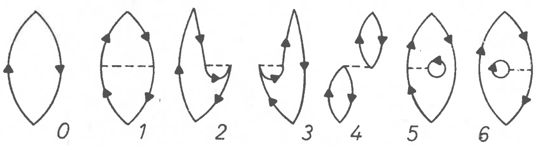

Fig. 3. The zeroth and first order polarization graphs. Graph *0* in the zeroth and graph *1* in the first order are characteristic graphs

0

$$M_{\mu\nu}(2\boldsymbol{k}, k_n) = -\left(\frac{e\hbar}{m}\right)^2 \int \frac{d^3p}{(2\pi)^3}\, p_\mu p_\nu (p+k,\, p-k)$$

$$M_{\mu\nu}(0, k_n) = 0.$$

In particular the longitudinal conductivity in the zeroth order is

$$\frac{k_\mu k_\nu}{k^2}\,\sigma^{(0)}_{\mu\nu}(\boldsymbol{k}, \omega) = \frac{ie^2 n}{\omega m} - \frac{i}{\omega}\,\frac{k_\mu k_\nu}{k^2}\, M^{+}_{\mu\nu}(\boldsymbol{k}, \omega) = \frac{i\omega e^2}{k^2}\, Q^{+}(\boldsymbol{k}, \hbar\omega).$$

Formulae for the contributions of the first order diagrams Fig. 3, No 1–4 are

1, 2, 3

$$M_{\mu\nu}(2\boldsymbol{k}, k_n) = \left(\frac{e\hbar}{m}\right)^2 \int \frac{d^3q}{(2\pi)^3}\,\frac{1}{\beta}\sum_m U(q, q_m) \int \frac{d^3p}{(2\pi)^3}\, K_{\mu\nu}(2\boldsymbol{k}, k_n)$$

1

$$K_{\mu\nu}(2\boldsymbol{k}, k_n) = \frac{p_\mu p_\nu - \dfrac{q_\mu q_\nu}{4}}{\left(p-k-\dfrac{q}{2}\right) - \left(p+k-\dfrac{q}{2}\right) + k_n}\;\frac{1}{\left(p-k+\dfrac{q}{2}\right) - \left(p+k+\dfrac{q}{2}\right) + k_n} \times$$

$$\times\left[\left(p-k-\frac{q}{2},\, p-k+\frac{q}{2}\right) - \left(p-k-\frac{q}{2},\, p+k+\frac{q}{2}\right) - \left(p+k-\frac{q}{2},\, p-k+\frac{q}{2}\right) + \right.$$

$$+\left(p+k-\frac{q}{2},\ p+k+\frac{q}{2}\right)\Big]$$

$$K_{\mu\nu}(0, k_n) = \frac{2}{k_n^2}\left(p_\mu p_\nu - \frac{q_\mu q_\nu}{4}\right)\left[\left(p-\frac{q}{2},\ p+\frac{q}{2}\right) - \left(p-\frac{q}{2},\ p+\frac{q}{2},\ k_n\right)\right]$$

2+3

$$K_{\mu\nu}(2k, k_n) = \frac{p_\mu p_\nu}{(p+k)-(p-k)-k_n}\Big\{(p+k,\ p+k+q)'-(p+k,\ p+k+q)_2-$$

$$-(p-k,\ p-k+q)'+(p-k,\ p-k+q)_2+\frac{1}{(p+k)-(p-k)-k_n}\big[(p+k,\ p-k+q)-$$

$$-(p+k,\ p+k+q)-(p-k,\ p-k+q)+(p-k,\ p+k+q)\big]\Big\}$$

$$K_{\mu\nu}(0, k_n) = -\frac{2}{k_n^2}\left(p_\mu p_\nu + \frac{q_\mu q_\nu}{4}\right)\left[\left(p-\frac{q}{2},\ p+\frac{q}{2}\right) - \left(p-\frac{q}{2},\ p+\frac{q}{2},\ k_n\right)\right]$$

4

$$M_{\mu\nu}(2k, k_n) = -\left(\frac{e\hbar}{m}\right)^2 U(2k, k_n)\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\, p_\mu p'_\nu (p+k,\ p-k)\,(p'+k,\ p'-k)$$

$$M_{\mu\nu}(0, k_n) = 0.$$

Formulae foı the contributions of the second order diagrams with one Fermion loop, Fig. 4, No 1–4 and Fig. 5, No 15–25:

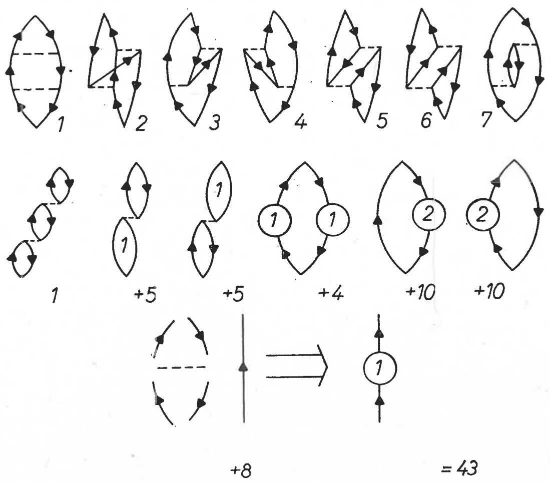

Fig. 4. The second order polarization graphs. Graphs *1* to *7* are characteristic graphs, the rest is pictured schematically (see also Fig. 5). (1) or (2) means here any first or second order graph of the single-particle Green's function, *(1)* means any first order polarization graph. Double arrow indicates replacement of the Fermion line by any first order (1) single-particle Green's function graph

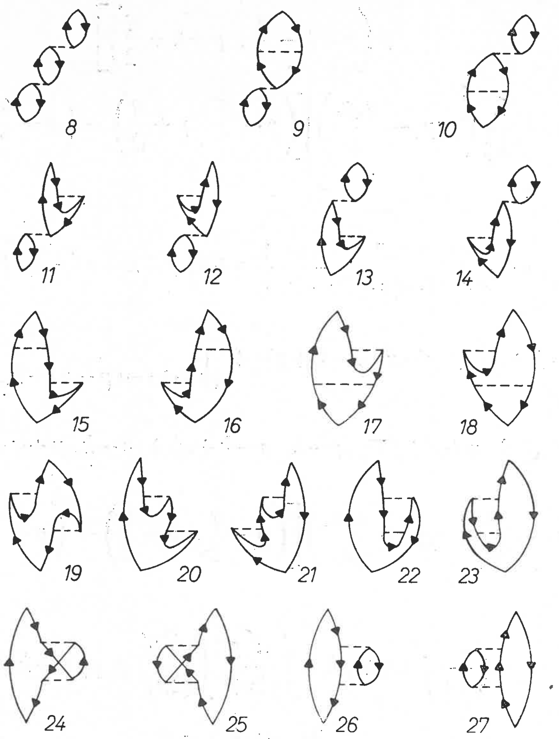

Fig. 5. The second order "uniform" polarization graphs. The characteristic graphs are omitted here, the numbering continues the numbering of characteristic graphs in Fig. 4

1−4, 15−25

$$M_{\mu\nu}(2\boldsymbol{k}, k_n) = -\left(\frac{e\hbar}{m}\right)^2 \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} \frac{1}{\beta^2} \sum_{m,m'} U(\boldsymbol{q}, q_m) \times$$

$$\times U(\boldsymbol{q}', q'_{m'}) \int \frac{d^3p}{(2\pi)^3} K_{\mu\nu}(2\boldsymbol{k}, k_n)$$

1+2

$$K_{\mu\nu}(2\boldsymbol{k}, k_n) = \frac{2\left(p_\mu p_\nu - \dfrac{(q_\mu + q'_\mu)(q_\nu + q'_\nu)}{4}\right)}{\left(p+k-\dfrac{q}{2}-\dfrac{q'}{2}\right) - \left(p-k-\dfrac{q}{2}-\dfrac{q'}{2}\right) - k_n} \times$$

$$\times \frac{1}{\left(p+k+\dfrac{q}{2}+\dfrac{q'}{2}\right) - \left(p-k+\dfrac{q}{2}+\dfrac{q'}{2}\right) - k_n} \times$$

$$\times\left\{\left[\frac{1}{\left(p+k-\frac{q}{2}-\frac{q'}{2}\right)-\left(p+k+\frac{q}{2}+\frac{q'}{2}\right)+q_m+q'_{m'}}-\right.\right.$$

$$\left.-\frac{1}{\left(p-k-\frac{q}{2}-\frac{q'}{2}\right)-\left(p+k+\frac{q}{2}+\frac{q'}{2}\right)+k_n+q_m+q'_{m'}}\right]\times$$

$$\times\left[\frac{\left(p+k-\frac{q}{2}-\frac{q'}{2},\ p+k+\frac{q}{2}-\frac{q'}{2}\right)-\left(p+k-\frac{q}{2}-\frac{q'}{2},\ p-k+\frac{q}{2}-\frac{q'}{2}\right)}{\left(p+k+\frac{q}{2}-\frac{q'}{2}\right)-\left(p-k+\frac{q}{2}-\frac{q'}{2}\right)-k_n}+\right.$$

$$\left.+\frac{\left(p+k-\frac{q}{2}-\frac{q'}{2},\ p+k-\frac{q}{2}+\frac{q'}{2}\right)-\left(p+k-\frac{q}{2}-\frac{q'}{2},\ p-k+\frac{q}{2}-\frac{q'}{2}\right)}{\left(p+k-\frac{q}{2}+\frac{q'}{2}\right)-\left(p-k+\frac{q}{2}-\frac{q'}{2}\right)-k_n+q_m-q'_{m'}}\right]+$$

$$+\left[\frac{1}{\left(p-k-\frac{q}{2}-\frac{q'}{2}\right)-\left(p-k+\frac{q}{2}+\frac{q'}{2}\right)+q_m+q'_{m'}}-\right.$$

$$\left.-\frac{1}{\left(p-k-\frac{q}{2}-\frac{q'}{2}\right)-\left(p+k+\frac{q}{2}+\frac{q'}{2}\right)+k_n+q_m+q'_{m'}}\right]\times$$

$$\times\left[\frac{\left(p-k-\frac{q}{2}-\frac{q'}{2},\ p+k+\frac{q}{2}-\frac{q'}{2}\right)-\left(p-k-\frac{q}{2}-\frac{q'}{2},\ p-k+\frac{q}{2}-\frac{q'}{2}\right)}{\left(p+k+\frac{q}{2}-\frac{q'}{2}\right)-\left(p-k+\frac{q}{2}-\frac{q'}{2}\right)-k_n}+\right.$$

$$\left.\left.+\frac{\left(p-k-\frac{q}{2}-\frac{q'}{2},\ p+k-\frac{q}{2}+\frac{q'}{3}\right)-\left(p-k-\frac{q}{2}-\frac{q'}{2},\ p-k+\frac{q}{2}-\frac{q'}{2}\right)}{\left(p+k-\frac{q}{2}+\frac{q'}{2}\right)-\left(p-k+\frac{q}{2}-\frac{q'}{2}\right)-k_n+q_m-q'_{m'}}\right]\right\}$$

$$K_{\mu\nu}(0,k_n)=\frac{2}{k_n^2}\left(p_\mu p_\nu-\frac{(q_\mu+q'_\mu)(q_\nu+q'_\nu)}{4}\right)\times$$

$$\times\left[\frac{1}{\left(p-\frac{q}{2}-\frac{q'}{2}\right)-\left(p+\frac{q}{2}+\frac{q'}{2}\right)+q_m+q'_{m'}}-\right.$$

$$- \frac{1}{\left(p-\frac{q}{2}-\frac{q'}{2}\right)-\left(p+\frac{q}{2}+\frac{q'}{2}\right)+k_n+q_m+q'_{m'}}\Bigg]\times$$

$$\times\left\{\frac{1}{k_n}\left[\left(p-\frac{q}{2}-\frac{q'}{2},\ p+\frac{q}{2}-\frac{q'}{2},\ -k_n\right)-\left(p-\frac{q}{2}-\frac{q'}{2},\ p+\frac{q}{2}-\frac{q'}{2}\right)-\right.\right.$$

$$\left.-\left(p-\frac{q}{2}-\frac{q'}{2},\ p+\frac{q}{2}-\frac{q'}{2},\ k_n\right)+\left(p-\frac{q}{2}-\frac{q'}{2},\ p+\frac{q}{2}-\frac{q'}{2}\right)\right]+$$

$$\left.+\frac{\left(p-\frac{q}{2}-\frac{q'}{2},\ p-\frac{q}{2}+\frac{q'}{2}\right)-\left(p-\frac{q}{2}-\frac{q'}{2},\ p+\frac{q}{2}-\frac{q'}{2},\ -k_n\right)+ +\left(p-\frac{q}{2}-\frac{q'}{2},\ p-\frac{q}{2}+\frac{q'}{2},\ k_n\right)-\left(p-\frac{q}{2}-\frac{q'}{2},\ p+\frac{q}{2}-\frac{q'}{2}\right)}{\left(p-\frac{q}{2}+\frac{q'}{2}\right)-\left(p+\frac{q}{2}-\frac{q'}{2}\right)-k_n+q_m-q'_{m'}}\right\}$$

3+4

$$K_{\mu\nu}(2\boldsymbol{k},k_n)=\frac{2\left(p_\mu p_\nu-\frac{q_\mu q_\nu}{4}\right)}{\left(p+k-\frac{q}{2}\right)-\left(p-k-\frac{q}{2}\right)-k_n}\ \frac{1}{\left(p+k+\frac{q}{2}\right)-\left(p-k+\frac{q}{2}\right)-k_n}\times$$

$$\times\left\{\frac{1}{\left(p-k+\frac{q}{2}+q'\right)-\left(p-k-\frac{q}{2}+q'\right)-q_m}\times\right.$$

$$\times\left[\frac{\left(p+k-\frac{q}{2},\ p-k+\frac{q}{2}+q'\right)-\left(p+k-\frac{q}{2},\ p-k-\frac{q}{2}+q'\right)}{\left(p+k-\frac{q}{2}\right)-\left(p+k+\frac{q}{2}\right)+q_m}+\right.$$

$$+\frac{\left(p-k-\frac{q}{2},\ p-k+\frac{q}{2}+q'\right)-\left(p-k-\frac{q}{2},\ p-k-\frac{q}{2}+q'\right)}{\left(p-k-\frac{q}{2}\right)-\left(p-k+\frac{q}{2}\right)+q_m}-$$

$$\left.-\frac{\left(p-k-\frac{q}{2},\ p-k+\frac{q}{2}+q'\right)-\left(p-k-\frac{q}{2},\ p-k-\frac{q}{2}+q'\right)- -\left(p+k+\frac{q}{2},\ p-k+\frac{q}{2}+q'\right)+\left(p+k+\frac{q}{2},\ p-k-\frac{q}{2}+q'\right)}{\left(p-k-\frac{q}{2}\right)-\left(p+k+\frac{q}{2}\right)+k_n+q_m}\right]+$$

$$+\frac{1}{\left(p+k+\frac{q}{2}+q'\right)-\left(p+k-\frac{q}{2}+q'\right)-q_m}\times$$

$$\times\left[\frac{\left(p-k-\frac{q}{2},\ p+k+\frac{q}{2}+q'\right)-\left(p-k-\frac{q}{2},\ p+k-\frac{q}{2}+q'\right)}{\left(p-k-\frac{q}{2}\right)-\left(p-k+\frac{q}{2}\right)+q_m}+\right.$$

$$+\frac{\left(p+k-\frac{q}{2},\ p+k+\frac{q}{2}+q'\right)-\left(p+k-\frac{q}{2},\ p+k-\frac{q}{2}+q'\right)}{\left(p+k-\frac{q}{2}\right)-\left(p+k+\frac{q}{2}\right)+q_m}-$$

$$-\left.\left.\frac{\left(p+k-\frac{q}{2},\ p+k+\frac{q}{2}+q'\right)-\left(p+k-\frac{q}{2},\ p+k-\frac{q}{2}+q'\right)-\left(p-k+\frac{q}{2},\ p+k+\frac{q}{2}+q'\right)+\left(p-k+\frac{q}{2},\ p+k-\frac{q}{2}+q'\right)}{\left(p+k-\frac{q}{2}\right)-\left(p-k+\frac{q}{2}\right)-k_n+q_m}\right]\right\}$$

$$K_{\mu\nu}(0,k_n)=\frac{1}{k_n^2}\,\frac{2\left(p_\mu p_\nu-\frac{q_\mu q_\nu}{4}\right)}{\left(p+\frac{q}{2}+q'\right)-\left(p-\frac{q}{2}+q'\right)+q_m}\left\{\frac{1}{\left(p-\frac{q}{2}\right)-\left(p+\frac{q}{2}\right)+q_m}\times\right.$$

$$\times\left[\left(p-\frac{q}{2},\ p+\frac{q}{2}+q',\ -k_n\right)+\left(p-\frac{q}{2},\ p+\frac{q}{2}+q',\ k_n\right)-\right.$$

$$-\left(p-\frac{q}{2},\ p-\frac{q}{2}+q',\ -k_n\right)-\left(p-\frac{q}{2},\ p-\frac{q}{2}+q',\ k_n\right)+$$

$$\left.+2\left(p-\frac{q}{2},\ p+\frac{q}{2}+q'\right)-2\left(p-\frac{q}{2},\ p-\frac{q}{2}+q'\right)\right]-$$

$$-\frac{\left(p-\frac{q}{2}\ \ p+\frac{q}{2}+q'\right)-\left(p-\frac{q}{2},\ p-\frac{q}{2}+q'\right)-\left(p+\frac{q}{2},\ p+\frac{q}{2}+q',\ -k_n\right)+\left(p+\frac{q}{2},\ p-\frac{q}{2}+q',\ -k_n\right)}{\left(p-\frac{q}{2}\right)-\left(p+\frac{q}{2}\right)+k_n+q_m}$$

$$\left(p-\frac{q}{2},\ p+\frac{q}{2}+q'\right)-\left(p-\frac{q}{2},\ p-\frac{q}{2}+q'\right)-$$

$$-\frac{-\left(p+\frac{q}{2},\ p+\frac{q}{2}+q',k_n\right)+\left(p+\frac{q}{2},\ p-\frac{q}{2}+q',k_n\right)}{\left(p-\frac{q}{2}\right)-\left(p+\frac{q}{2}\right)-k_n+q_m}\Bigg\}$$

15+16+17+18

$$K_{\mu\nu}(2\boldsymbol{k},k_n)=\frac{2\left(p_\mu p_\nu-\frac{q_\mu q_\nu}{4}\right)}{\left(p+k-\frac{q}{2}\right)-\left(p-k+\frac{q}{2}\right)-k_n}\ \frac{1}{\left(p+k+\frac{q}{2}\right)-\left(p-k+\frac{q}{2}\right)-k_n}\times$$

$$\times\left\{\frac{1}{\left(p-k-\frac{q}{2}+q'\right)-\left(p-k-\frac{q}{2}\right)-q'_{m'}}\left[\left(p-k-\frac{q}{2},\ p+k+\frac{q}{2}\right)'-\right.\right.$$

$$-\left(p-k-\frac{q}{2},\ p+k+\frac{q}{2}\right)_2-\left(p-k-\frac{q}{2},\ p-k+\frac{q}{2}\right)'+\left(p-k-\frac{q}{2},\ p-k+\frac{q}{2}\right)_2+$$

$$\left(p+k-\frac{q}{2},\ p+k+\frac{q}{2}\right)-\left(p+k-\frac{q}{2},\ p-k+\frac{q}{2}\right)-$$

$$+\frac{-\left(p-k-\frac{q}{2}+q',\ p+k+\frac{q}{2}\right)+\left(p-k-\frac{q}{2}+q',\ p-k+\frac{q}{2}\right)}{\left(p+k-\frac{q}{2}\right)-\left(p-k-\frac{q}{2}+q'\right)-k_n+q'_{m'}}$$

$$\left(p-k-\frac{q}{2},\ p+k+\frac{q}{2}\right)-\left(p-k-\frac{q}{2},\ p-k+\frac{q}{2}\right)-$$

$$\left.-\frac{-\left(p-k-\frac{q}{2}+q',\ p+k+\frac{q}{2}\right)+\left(p-k-\frac{q}{2}+q',\ p-k+\frac{q}{2}\right)}{\left(p-k-\frac{q}{2}\right)-\left(p-k-\frac{q}{2}+q'\right)+q'_{m'}}\right]+$$

$$+\frac{1}{\left(p+k-\frac{q}{2}+q'\right)-\left(p+k-\frac{q}{2}\right)-q'_{m'}}\left[\left(p+k-\frac{q}{2},\ p-k+\frac{q}{2}\right)'-\right.$$

$$-\left(p+k-\frac{q}{2},\ p-k+\frac{q}{2}\right)_2 - \left(p+k-\frac{q}{2},\ p+k+\frac{q}{2}\right)' + \left(p+k-\frac{q}{2},\ p+k+\frac{q}{2}\right)_2 +$$

$$+\frac{\left(p-k-\frac{q}{2},\ p-k+\frac{q}{2}\right) - \left(p-k-\frac{q}{2},\ p+k+\frac{q}{2}\right) - \left(p+k-\frac{q}{2}+q',\ p-k+\frac{q}{2}\right) + \left(p+k-\frac{q}{2}+q',\ p+k+\frac{q}{2}\right)}{\left(p-k-\frac{q}{2}\right) - \left(p+k-\frac{q}{2}+q'\right) + k_n + q'_{m'}} -$$

$$\left.-\frac{\left(p+k-\frac{q}{2},\ p-k+\frac{q}{2}\right) - \left(p+k-\frac{q}{2},\ p+k+\frac{q}{2}\right) - \left(p+k-\frac{q}{2}+q',\ p-k+\frac{q}{2}\right) + \left(p+k-\frac{q}{2}+q',\ p+k+\frac{q}{2}\right)}{\left(p+k-\frac{q}{2}\right) - \left(p+k-\frac{q}{2}+q'\right) + q'_{m'}}\right] +$$

$$+\left[\frac{1}{\left(p+k-\frac{q}{2}+q'\right) - \left(p+k-\frac{q}{2}\right) - q'_{m'}} - \frac{1}{\left(p-k-\frac{q}{2}+q'\right) - \left(p-k-\frac{q}{2}\right) - q'_{m'}}\right] \times$$

$$\left.\times \frac{\left(p+k-\frac{q}{2},\ p+k+\frac{q}{2}\right) - \left(p+k-\frac{q}{2},\ p-k+\frac{q}{2}\right) - \left(p-k-\frac{q}{2},\ p+k+\frac{q}{2}\right) + \left(p-k-\frac{q}{2},\ p-k+\frac{q}{2}\right)}{\left(p+k-\frac{q}{2}\right) - \left(p-k-\frac{q}{2}\right) - k_n}\right\}$$

$$K_{\mu\nu}(0, k_n) = \frac{1}{k_n^2} \frac{2\left(p_\mu p_\nu - \frac{q_\mu q_\nu}{4}\right)}{\left(p-\frac{q}{2}+q'\right) - \left(p-\frac{q}{2}\right) - q'_{m'}} \left[\left(p-\frac{q}{2},\ p+\frac{q}{2},\ k_n\right)' -\right.$$

$$-\left(p-\frac{q}{2},\ p+\frac{q}{2},\ k_n\right)_2 + \left(p-\frac{q}{2},\ p+\frac{q}{2},\ -k_n\right)' - \left(p-\frac{q}{2},\ p+\frac{q}{2},\ -k_n\right)_2 -$$

$$-2\left(p-\frac{q}{2},\ p+\frac{q}{2}\right)' + 2\left(p-\frac{q}{2},\ p+\frac{q}{2}\right)_2 +$$

$$+\frac{\left(p-\frac{q}{2},\ p+\frac{q}{2}\right)-\left(p-\frac{q}{2},\ p+\frac{q}{2},\ -k_n\right)-\\-\left(p-\frac{q}{2}+q',\ p+\frac{q}{2},\ k_n\right)+\left(p-\frac{q}{2}+q',\ p+\frac{q}{2}\right)}{\left(p-\frac{q}{2}\right)-\left(p-\frac{q}{2}+q'\right)-k_n+q'_{m'}}+$$

$$+\frac{\left(p-\frac{q}{2},\ p+\frac{q}{2}\right)-\left(p-\frac{q}{2},\ p+\frac{q}{2},\ k_n\right)-\\-\left(p-\frac{q}{2}+q',\ p+\frac{q}{2},\ -k_n\right)+\left(p-\frac{q}{2}+q',\ p+\frac{q}{2}\right)}{\left(p-\frac{q}{2}\right)-\left(p-\frac{q}{2}+q'\right)+k_n+q'_{m'}}-$$

$$-\frac{\left(p-\frac{q}{2},\ p+\frac{q}{2},\ k_n\right)+\left(p-\frac{q}{2},\ p+\frac{q}{2},\ -k_n\right)-\left(p-\frac{q}{2}+q',\ p+\frac{q}{2},\ k_n\right)-\\-\left(p-\frac{q}{2}+q',\ p+\frac{q}{2},\ -k_n\right)-2\left(p-\frac{q}{2},\ p+\frac{q}{2}\right)}{\left(p-\frac{q}{2}\right)-\left(p-\frac{q}{2}+q'\right)+q'_{m'}}+$$

$$+\left.\frac{2\left(p-\frac{q}{2}+q',\ p+\frac{q}{2}\right)}{\left(p-\frac{q}{2}\right)-\left(p-\frac{q}{2}+q'\right)+q'_{m'}}\right]$$

19

$$K_{\mu\nu}(2\boldsymbol{k},\ k_n)=\frac{p_\mu p_\nu}{[(p+k)-(p-k)-k_n]^2}\left\{\frac{(p+k,\ p+k+q')'}{(p+k)-(p-k+q)-k_n+q_m}+\right.$$

$$+\frac{(p-k,\ p+k+q')'}{(p-k)-(p-k+q)+q_m}+\frac{1}{(p-k+q)-(p+k+q')+k_n-q_m+q'_{m'}}\times$$

$$\times\left[(p+k,\ p+k+q')_2-(p+k,\ p-k+q)_2+(p-k,\ p+k+q')_2-(p-k,\ p-k+q)_2-\right.$$

$$\left.\left.-2\frac{(p+k,\ p-k+q)-(p+k,\ p+k+q')-(p-k,\ p-k+q)+(p-k,\ p+k+q')}{(p+k)-(p-k)-k_n}\right]\right\}$$

$$K_{\mu\nu}(0,\ k_n)=\frac{p_\mu p_\nu}{k_n^2}\left\{\frac{(p,\ p+q')'}{(p)-(p+q)-k_n+q_m}+\frac{(p,\ p+q',\ k_n)'}{(p)-(p+q)+q_m}+\right.$$

$$+\frac{1}{(p+q)-(p+q')+k_n-q_m+q'_{m'}}\Big[(p,\,p+q')_2-(p,\,p+q,\,-k_n)_2+$$

$$+(p,\,p+q',\,k_n)_2-(p,\,p+q)_2+\frac{2}{k_n}\,[(p,\,p+q,\,-k_n)-(p,\,p+q')-(p,\,p+q)+$$

$$+(p,\,p+q',\,k_n)]\Big]\Big\}$$

20+21

$$K_{\mu\nu}(2\boldsymbol{k},\,k_n)=2p_\mu p_\nu\Big\{\frac{1}{(p-k+q)-(p-k+q')-q_m+q'_{m'}}\Big[\Big[\frac{1}{(p+k)-(p-k)-k_n}+$$

$$+\frac{1}{(p+k)-(p-k)+k_n}\Big][(p-k,\,p-k+q)'_2-(p-k,\,p-k+q)''-(p-k,\,p-k+q)_3]-$$

$$-\Big[\frac{1}{[(p+k)-(p-k)-k_n]^2}+\frac{1}{[(p+k)-(p-k)+k_n]^2}\Big][(p-k,\,p-k+q)'-$$

$$-(p-k,\,p-k+q)_2]-\Big[\frac{1}{[(p+k)-(p-k)-k_n]^3}+$$

$$+\frac{1}{[(p+k)-(p-k)+k_n]^3}\Big](p-k,\,p-k+q)\Big]+\frac{1}{[(p+k)-(p-k)-k_n]^3}\times$$

$$\times\Big[\frac{(p+k,\,p-k+q)}{(p-k+q)-(p-k+q')-q_m+q'_{m'}}-\frac{(p-k,\,p+k+q)}{(p+k+q)-(p+k+q')-q_m+q'_{m'}}\Big]\Big\}$$

$$K_{\mu\nu}(0,\,k_n)=-\frac{4p_\mu p_\nu}{k_n^2}\,\frac{1}{(p+q)-(p+q')-q_m+q'_{m'}}\Big[(p,\,p+q)'-(p,\,p+q)_2+$$

$$+\frac{(p,\,p+q,\,-k_n)-(p,\,p+q,\,k_n)}{2k_n}\Big]$$

22+23

$$K_{\mu\nu}(2\boldsymbol{k},\,k_n)=p_\mu p_\nu\Big\{\Big[\frac{1}{(p+k)-(p-k)-k_n}+\frac{1}{(p+k)-(p-k)+k_n}\Big]\times$$

$$\times\Big[\frac{(p-k+q,\,p-k+q+q')_2-(p-k+q,\,p-k+q+q')'+(p-k,\,p-k+q+q')_2-(p-k,\,p-k+q+q')'}{[(p-k)-(p-k+q)+q_m]^2}+$$

$$+2\,\frac{(p-k,\,p-k+q+q')-(p-k+q,\,p-k+q+q')}{[(p-k)-(p-k+q)+q_m]^3}\Big]-$$

$$-\Big[\frac{1}{[(p+k)-(p-k)-k_n]^2}+\frac{1}{[(p+k)-(p-k)+k_n]^2}\Big]\times$$

$$\times\left[\frac{(p-k,\ p-k+q+q')-(p-k+q,\ p-k+q+q')}{[(p-k)-(p-k+q)+q_m]^2}+\right.$$

$$\left.+\frac{(p-k+q,\ p-k+q+q')_2-(p-k+q,\ p-k+q+q')'}{(p-k)-(p-k+q)+q_m}\right]+$$

$$+\frac{1}{[(p+k)-(p-k)-k_n]^2}\left[\frac{(p+k,\ p-k+q+q')-(p-k+q,\ p-k+q+q')}{[(p+k)-(p-k+q)-k_n+q_m]^2}+\right.$$

$$+\frac{(p-k+q,\ p-k+q+q')_2-(p-k+q,\ p-k+q+q')'}{(p+k)-(p-k+q)-k_n+q_m}+$$

$$+\frac{(p-k,\ p+k+q+q')-(p+k+q,\ p+k+q+q')}{[(p-k)-(p+k+q)+k_n+q_m]^2}+$$

$$\left.\left.+\frac{(p+k+q,\ p+k+q+q')_2-(p+k+q,\ p+k+q+q')'}{(p-k)-(p+k+q)+k_n+q_m}\right]\right\}$$

$$K_{\mu\nu}(0,k_n)=\frac{p_\mu p_\nu}{k_n^2}\left\{\frac{(p,\ p+q+q',\ -k_n)-(p+q,\ p+q+q')}{[(p)-(p+q)-k_n+q_m]^2}+\right.$$

$$+\frac{(p,\ p+q+q',\ k_n)-(p+q,\ p+q+q')}{[(p)-(p+q)+k_n+q_m]^2}+\left[\frac{1}{(p)-(p+q)-k_n+q_m}+\right.$$

$$\left.+\frac{1}{(p)-(p+q)+k_n+q_m}\right][(p+q,\ p+q+q')_2-(p+q,\ p+q+q')']-$$

$$\left.-2\left[\frac{(p,\ p+q+q')-(p+q,\ p+q+q')}{[(p)-(p+q)+q_m]^2}+\frac{(p+q,\ p+q+q')_2-(p+q,\ p+q+q')'}{(p)-(p+q)+q_m}\right]\right\}$$

24+25

$$K_{\mu\nu}(2\boldsymbol{k},k_n)=2p_\mu p_\nu\left\{\frac{1}{(p-k+q)-(p-k+q')-q_m+q'_m}\times\right.$$

$$\times\left[\left[\frac{1}{(p+k)-(p-k)-k_n}+\frac{1}{(p+k)-(p-k)+k_n}\right]\times\right.$$

$$\times\frac{(p-k,\ p-k+q)'-(p-k,\ p-k+q)_2-(p-k,\ p-k+q+q')'+(p-k,\ p-k+q+q')_2}{(p-k+q+q')-(p-k+q)'-q'_{m'}}+$$

$$+\left[\frac{1}{[(p+k)-(p-k)-k_n]^2}+\frac{1}{[(p+k)-(p-k)+k_n]^2}\right]\times$$

$$\times\frac{(p-k+q,\ p-k+q+q')-(p-k,\ p-k+q+q')}{(p-k)-(p-k+q)+q_m}+$$

$$\left.+\frac{1}{[(p+k)-(p-k)-k_n]^2}\ \frac{(p+k,\ p-k+q+q')-(p-k+q,\ p-k+q+q')}{(p+k)-(p-k+q)-k_n+q_m}\right]+$$

$$+\frac{1}{[(p+k)-(p-k)-k_n]^2}\,\frac{1}{(p+k+q)-(p+k+q')-q_m+q'_m}\times$$

$$\times\frac{(p-k,\ p+k+q+q')-(p+k+q,\ p+k+q+q')}{(p-k)-(p+k+q)+k_n+q_m}\Bigg\}$$

$$K_{\mu\nu}(0,k_n)=\frac{2}{k_n^2}\,\frac{1}{(p+q)-(p+q')-q_m+q'_m}\left[2\,\frac{(p+q,\ p+q+q')-(p,\ p+q+q')}{(p)-(p+q)+q_m}+\right.$$

$$\left.+\frac{(p,\ p+q+q',\ -k_n)-(p+q,\ p+q+q')}{(p)-(p+q)-k_n+q_m}+\frac{(p,\ p+q+q',\ k_n)-(p+q,\ p+q+q')}{(p)-(p+q)+k_n+q_m}\right].$$

Contributions of the second order graphs with two Fermion loops, Fig. 4 and Fig. 5, No 5–7, 9–14, 26, 27 are

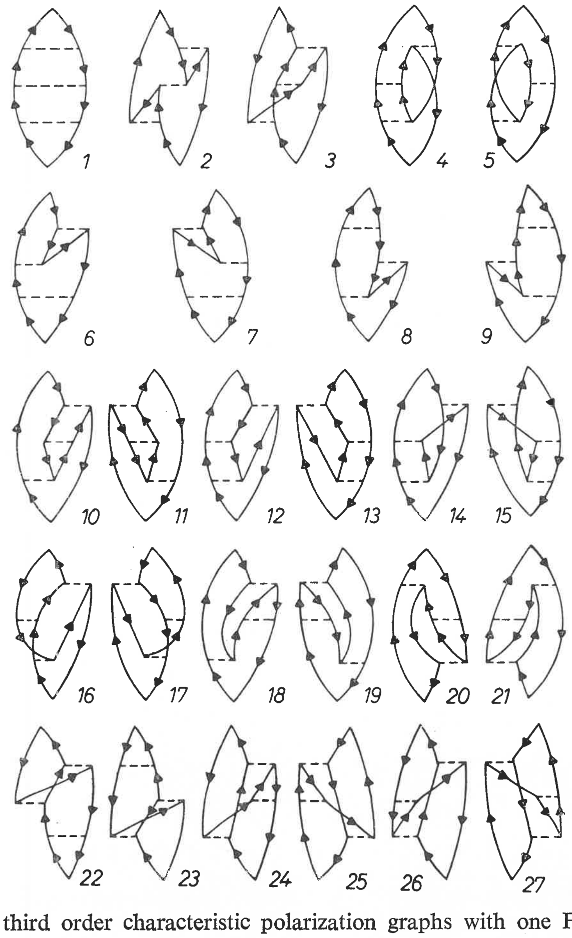

Fig. 6. The third order characteristic polarization graphs with one Fermion loop

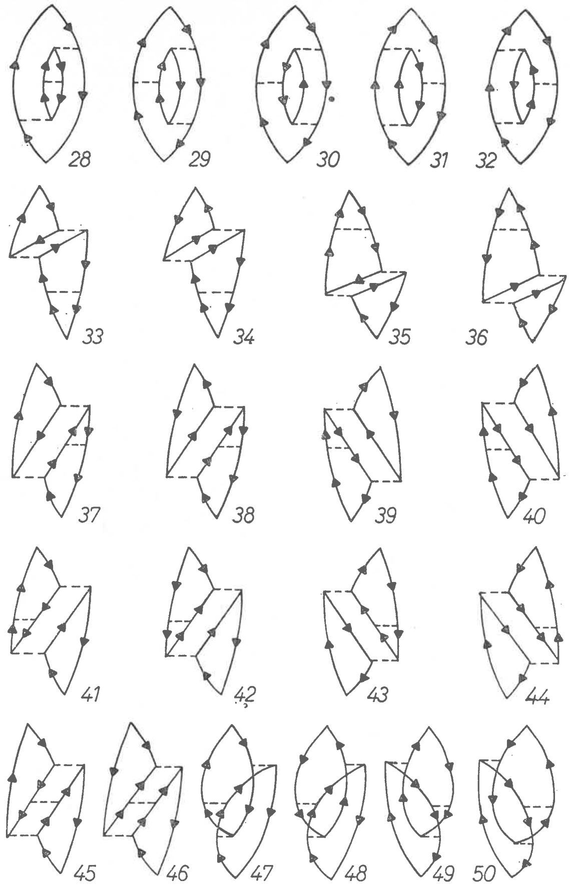

Fig. 7. The third order characteristic polarization graphs with two Fermion loops (see also Fig. 8)

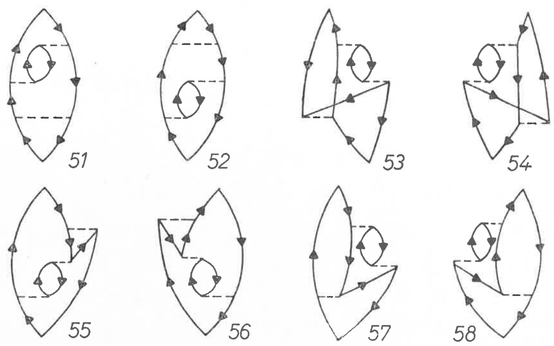

Fig. 8. The third order characteristic polarization graphs with two Fermion loops. One loop modifies the interaction line (see also Fig. 7)

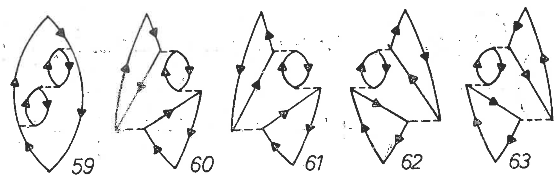

Fig. 9. The third order characteristic polarization graphs with three Fermion loops

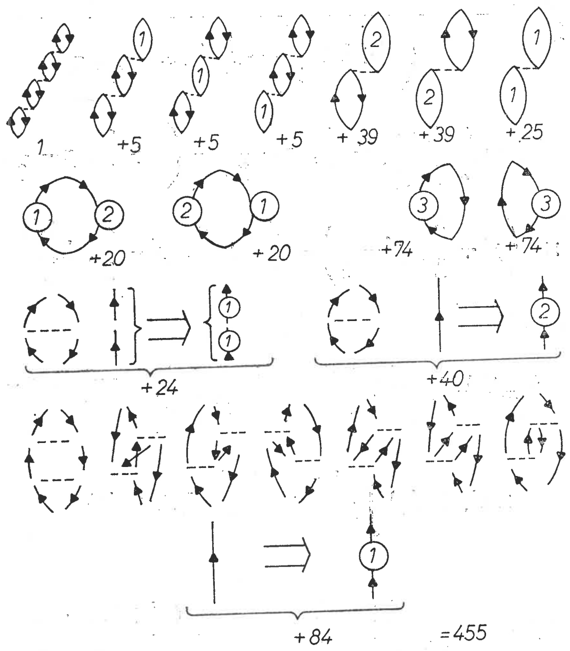

Fig. 10. The third order polarization graphs pictured schematically. The characteristic graphs are omitted here. ①, ②, ③ mean any first, second or third order graph respectively of the single-particle Green's function, *(1)* or *(2)* pictures any first or second order polarization graph. Double arrows indicate replacement of the Fermion lines by the first ① or second ② order single-particle Green's function graphs

5+6

$$M_{\mu\nu}(2\boldsymbol{k}, k_n) = \left(\frac{e\hbar}{m}\right)^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{\beta} \sum_m U(\boldsymbol{q}, q_m) U(2\boldsymbol{k}+\boldsymbol{q}, k_n + q_m) \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} K_{\mu\nu}(2\boldsymbol{k}, k_n)$$

$$K_{\mu\nu}(2k, k_n) = \left(p_\mu - \frac{q_\mu}{2}\right) \frac{\left(p+k-\frac{q}{2},\, p+k+\frac{q}{2}\right) - \left(p-k-\frac{q}{2},\, p+k+\frac{q}{2}\right)}{\left(p+k-\frac{q}{2}\right) - \left(p-k-\frac{q}{2}\right) - k_n} \times$$

$$\times\left[\left(p'_\nu-\frac{q_\nu}{2}\right)\frac{\left(p'+k-\frac{q}{2},\ p'+k+\frac{q}{2}\right)-\left(p'-k-\frac{q}{2},\ p'+k+\frac{q}{2}\right)}{\left(p'+k-\frac{q}{2}\right)-\left(p'-k-\frac{q}{2}\right)-k_n}+\right.$$

$$\left.+\left(p'_\nu+\frac{q_\nu}{2}\right)\frac{\left(p'+k+\frac{q}{2},\ p'-k-\frac{q}{2}\right)-\left(p'-k+\frac{q}{2},\ p'-k-\frac{q}{2}\right)}{\left(p'+k+\frac{q}{2}\right)-\left(p'+k+\frac{q}{2}\right)-k_n}\right]$$

$$K_{\mu\nu}(0,k_n)=\frac{q_\mu q_\nu}{2k_n^2}\left[\left(p+\frac{q}{2},\ p-\frac{q}{2}\right)-\left(p+\frac{q}{2},\ p-\frac{q}{2},\ k_n\right)\right]\times$$

$$\times\left[\left(p'+\frac{q}{2},\ p'-\frac{q}{2}\right)-\left(p'+\frac{q}{2},\ p'-\frac{q}{2},\ k_n\right)\right]$$

9, 10, 11, 12, 13, 14

$$M_{\mu\nu}(2\boldsymbol{k},k_n)=\left(\frac{e\hbar}{m}\right)^2\int\frac{d^3q}{(2\pi)^3}\,\frac{1}{\beta}\sum_m U(\boldsymbol{q},q_m)\times$$

$$\times U(2k,k_n)\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}K_{\mu\nu}(2\boldsymbol{k},k_n)$$

9+10

$$K_{\mu\nu}(2\boldsymbol{k},k_n)=\frac{p_\mu p'_\nu+p_\nu p'_\mu}{(p+k)-(p-k)-k_n}\ \frac{(p'+k,\ p'-k)}{(p+k+q)-(p-k+q)-k_n}\times$$

$$\times[(p+k,\ p+k+q)-(p+k,\ p-k+q)-(p-k,\ p+k+q)+(p-k,\ p-k+q)]$$

$$K_{\mu\nu}(0,k_n)=0$$

11+12+13+14

$$K_{\mu\nu}(2\boldsymbol{k},k_n)=(p_\mu p'_\nu+p_\nu p'_\mu)\,\frac{(p'+k,\ p'-k)}{(p+k)-(p-k)-k_n}\Big\{(p+k,\ p+k+q)'-$$

$$-(p+k,\ p+k+q)_2-(p-k,\ p-k+q)'+(p-k,\ p-k+q)_2+$$

$$+\frac{1}{(p+k)-(p-k)-k_n}[(p+k,\ p-k+q)-(p+k,\ p+k+q)-$$

$$-(p-k,\ p-k+q)+(p-k,\ p+k+q)]\Big\}$$

$$K_{\mu\nu}(0,k_n)=0$$

7, 26, 27

$$M_{\mu\nu}(2\boldsymbol{k}, k_n) = \left(\frac{e\hbar}{m}\right)^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\beta} \sum_m [U(\boldsymbol{q}, q_m)]^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} K_{\mu\nu}(2\boldsymbol{k}, k_n)$$

7

$$K_{\mu\nu}(2\boldsymbol{k}, k_n) = \frac{p_\mu p_\nu - \dfrac{q_\mu q_\nu}{4}}{\left(p-k-\dfrac{q}{2}\right) - \left(p+k-\dfrac{q}{2}\right) + k_n} \, \frac{(p'+q, p')}{\left(p-k+\dfrac{q}{2}\right) - \left(p+k+\dfrac{q}{2}\right) + k_n} \times$$

$$\times \left[\left(p-k-\frac{q}{2}, p-k+\frac{q}{2}\right) - \left(p-k-\frac{q}{2}, p+k+\frac{q}{2}\right) - \right.$$

$$\left. - \left(p+k-\frac{q}{2}, p-k+\frac{q}{2}\right) + \left(p+k-\frac{q}{2}, p+k+\frac{q}{2}\right)\right]$$

$$K_{\mu\nu}(0, k_n) = \frac{2}{k_n^2}\left(p_\mu p_\nu - \frac{q_\mu q_\nu}{4}\right)(p'+q, p')\left[\left(p-\frac{q}{2}, p+\frac{q}{2}\right) - \right.$$

$$\left. - \left(p-\frac{q}{2}, p+\frac{q}{2}, k_n\right)\right]$$

26+27

$$K_{\mu\nu}(2\boldsymbol{k}, k_n) = p_\mu p_\nu \frac{(p'+q, p')}{(p+k)-(p-k)-k_n} \Big\{(p+k, p+k+q)' - (p+k, p+k+q)_2 -$$

$$-(p-k, p-k+q)' + (p-k, p-k+q)_2 + \frac{1}{(p+k)-(p-k)-k_n} [(p+k, p-k+q) -$$

$$-(p+k, p+k+q) - (p-k, p-k+q) + (p-k, p+k+q)]\Big\}$$

$$K_{\mu\nu}(0, k_n) = -\frac{2}{k_n^2}\left(p_\mu p_\nu + \frac{q_\mu q_\nu}{4}\right)(p'+q, p')\left[\left(p-\frac{q}{2}, p+\frac{q}{2}\right) - \right.$$

$$\left. - \left(p-\frac{q}{2}, p+\frac{q}{2}, k_n\right)\right].$$

Contribution of the second order graph with three Fermion loops, Fig. 5, No 8:

8

$$M_{\mu\nu}(2\boldsymbol{k}, k_n) = -\left(\frac{e\hbar}{m}\right)^2 [U(2\boldsymbol{k}, k_n)]^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3p''}{(2\pi)^3} p_\mu p''_\nu \times$$

$$\times (p+k, p-k)(p'+k, p'-k)(p''+k, p''-k)$$

$$M_{\mu\nu}(0, k_n) = 0$$

We add as an example formulae for a set of eight particular third order diagrams exhibited in Fig. 7.

Contributions of the sum of the eight diagrams of the third order Fig. 7, No 37–44:

37+38+39+40+41+42+43+44

$$M_{\mu\nu}(2\boldsymbol{k}, k_n) = -\left(\frac{\hbar e}{m}\right)^2 \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} \frac{1}{\beta^2} \sum_{m,m'} U(\boldsymbol{q}, q_m) U(2\boldsymbol{k}+\boldsymbol{q}, k_n+q_m) \times$$

$$\times U(\boldsymbol{q}', q'_{m'}) \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} K_{\mu\nu}(2\boldsymbol{k}, k_n)$$

$$K_{\mu\nu}(2\boldsymbol{k}, k_n) = \left\{\left[\left(p_\mu - \frac{q_\mu}{2}\right)\left(p'_\nu - \frac{q_\nu}{2}\right) + \left(p_\nu - \frac{q_\nu}{2}\right)\left(p'_\mu - \frac{q_\mu}{2}\right)\right] \times\right.$$

$$\times \frac{\left(p'-k-\frac{q}{2},\ p'+k+\frac{q}{2}\right) - \left(p'+k-\frac{q}{2},\ p'+k+\frac{q}{2}\right)}{\left(p'-k-\frac{q}{2}\right) - \left(p'+k-\frac{q}{2}\right) + k_n} +$$

$$+\left[\left(p_\mu - \frac{q_\mu}{2}\right)\left(p'_\nu - \frac{q_\nu}{2}\right) + \left(p_\nu - \frac{q_\nu}{2}\right)\left(p'_\mu + \frac{q_\mu}{2}\right)\right] \times$$

$$\left. \times \frac{\left(p'-k+\frac{q}{2},\ p'-k-\frac{q}{2}\right) - \left(p'+k+\frac{q}{2},\ p'-k-\frac{q}{2}\right)}{\left(p'-k+\frac{q}{2}\right) - \left(p'+k+\frac{q}{2}\right) + k_n} \right\} \times$$

$$\times \frac{1}{\left(p-k-\frac{q}{2}\right) - \left(p+k-\frac{q}{2}\right) + k_n} \frac{1}{\left(p+k+\frac{q}{2}\right) - \left(p+k+\frac{q}{2}+q'\right) + q'_{m'}} \times$$

$$\times \left[\frac{\left(p-k-\frac{q}{2},\ p+k-\frac{q}{2}+q'\right) - \left(p+k+\frac{q}{2},\ p+k-\frac{q}{2}+q'\right) + \left(p-k-\frac{q}{2},\ p-k-\frac{q}{2}+q'\right) - \left(p+k+\frac{q}{2},\ p-k-\frac{q}{2}+q'\right)}{\left(p-k-\frac{q}{2}\right) - \left(p+k+\frac{q}{2}\right) + k_n + q_m} - \right.$$

$$+\frac{\left(p-k-\frac{q}{2},\ p+k-\frac{q}{2}+q'\right)-\left(p+k+\frac{q}{2}+q',\ p+k-\frac{q}{2}+q'\right)+ +\left(p-k-\frac{q}{2},\ p-k-\frac{q}{2}+q'\right)-\left(p+k+\frac{q}{2}+q',\ p-k-\frac{q}{2}+q'\right)}{\left(p-k-\frac{q}{2}\right)-\left(p+k+\frac{q}{2}+q'\right)+k_n+q_m+q'_{m'}}-$$

$$-\frac{\left(p+k-\frac{q}{2},\ p+k-\frac{q}{2}+q'\right)-\left(p+k+\frac{q}{2},\ p+k-\frac{q}{2}+q'\right)+ +\left(p+k-\frac{q}{2},\ p-k-\frac{q}{2}+q'\right)-\left(p+k+\frac{q}{2},\ p-k-\frac{q}{2}+q'\right)}{\left(p+k-\frac{q}{2}\right)-\left(p+k+\frac{q}{2}\right)+q_m}+$$

$$\left.+\frac{\left(p+k-\frac{q}{2},\ p+k-\frac{q}{2}+q'\right)-\left(p+k+\frac{q}{2}+q',\ p+k-\frac{q}{2}+q'\right)+ +\left(p+k-\frac{q}{2},\ p-k-\frac{q}{2}+q'\right)-\left(p+k+\frac{q}{2}+q',\ p-k-\frac{q}{2}+q'\right)}{\left(p+k-\frac{q}{2}\right)-\left(p+k+\frac{q}{2}+q'\right)+q_m+q'_{m'}}\right]$$

$$K_{\mu\nu}(0,k_n)=\frac{2}{k_n^2}\left[\left(p_\mu-\frac{q_\mu}{2}\right)p'_\nu+\left(p_\nu-\frac{q_\nu}{2}\right)p'_\mu\right]\times$$

$$\times\frac{\left(p'-\frac{q}{2},\ p'+\frac{q}{2},\ k_n\right)-\left(p'-\frac{q}{2},\ p'+\frac{q}{2}\right)}{\left(p+\frac{q}{2}\right)-\left(p+\frac{q}{2}+q'\right)+q'_{m'}}\times$$

$$\times\left[\frac{\left(p-\frac{q}{2},\ p-\frac{q}{2}+q',k_n\right)-\left(p+\frac{q}{2},\ p-\frac{q}{2}+q'\right)+ +\left(p-\frac{q}{2},\ p-\frac{q}{2}+q'\right)-\left(p+\frac{q}{2},\ p-\frac{q}{2}+q',-k_n\right)}{\left(p-\frac{q}{2}\right)-\left(p+\frac{q}{2}\right)+k_n+q_m}-\right.$$

$$-\frac{\left(p-\frac{q}{2},\,p-\frac{q}{2}+q',\,k_n\right)-\left(p+\frac{q}{2}+q',\,p-\frac{q}{2}+q'\right)+\left(p-\frac{q}{2},\,p-\frac{q}{2}+q'\right)-\left(p+\frac{q}{2}+q',\,p-\frac{q}{2}+q',\,-k_n\right)}{\left(p-\frac{q}{2}\right)-\left(p+\frac{q}{2}+q'\right)+k_n+q_m+q'_{m'}}-$$

$$-\frac{\left(p-\frac{q}{2},\,p-\frac{q}{2}+q'\right)-\left(p+\frac{q}{2},\,p-\frac{q}{2}+q'\right)+\left(p-\frac{q}{2},\,p-\frac{q}{2}+q',\,-k_n\right)-\left(p+\frac{q}{2},\,p-\frac{q}{2}+q',\,-k_n\right)}{\left(p-\frac{q}{2}\right)-\left(p+\frac{q}{2}\right)+q_m}+$$

$$+\frac{\left(p-\frac{q}{2},\,p-\frac{q}{2}+q'\right)-\left(p+\frac{q}{2}+q',\,p-\frac{q}{2}+q'\right)+\left(p-\frac{q}{2},\,p-\frac{q}{2}+q',\,-k_n\right)-\left(p+\frac{q}{2}+q',\,p-\frac{q}{2}+q',\,-k_n\right)}{\left(p-\frac{q}{2}\right)-\left(p+\frac{q}{2}+q'\right)+q_m+q'_{m'}}\Bigg]$$

Al the first order and two of the second order polarization diagrams, No 5 and 6 of Fig. 4, have been recently evaluated for the zero frequency by Geldart and Taylor [11] who performed numerically the necessary multi-dimensional integrations.

The integrations over the momentum variables cannot be performed analytically except for very particular approximations like the logarithmic approximation used extensively in the theory of Kondo effect [26–29] and in the description of the Mahan singularities [18].

4. *Conclusions*

The enumeration of diagrams given above can be of practical value, especially in view of recent development of numerical techniques which will allow eventually explicit evaluation of the lowest order diagrams for the electron gas in three dimensions. This formidable task requires a large amount of computation labour but it is a finite task within finite order of the perturbation [18]. The complete enumeration of the third-order polarization and Green's functions diagrams we could not find in the literature. We do not write here the expressions corresponding to all the diagrams of the third order, but once the diagrams have all been presented, expressions for them can be written down according to known rules [1, 3, 16, 24, 25]. Examination of the self-consistency criteria [1, 9, 10] which *i. al.*

retain the conservation laws requires also an insight into the higher order graphs. Discussion of the conjugate graphs whose contributions added together simplify or cancel at particular limits [5–7, 9–11, 17, 18, 20, 21], demands also, if it has to be extended beyond the second order in the interaction, enumeration of the next order, that is the third order graphs.

Towards these objectives our enumeration constitutes a first step. The formulae for the contributions of the lowest order diagram can help to assess the amount of labour necessary to evaluate them.

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