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SPIN WAVE THEORY OF TWO-SUBLATTICE UNIAXIAL HEISENBERG FERRIMAGNETS WITH EXTERNAL MAGNETIC FIELD. II. THERMODYNAMICAL PROPERTIES

By L. Biegała

Institute for Low Temperature and Structure Research, Polish Academy od Sciences, Wrocław*

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In Part I of this paper the spin-wave theory was applied to a two-sublattice uniaxial Heisenberg ferrimagnet in an external magnetic field parallel or perpendicular to the anisotropy axis. In the free-particle approximation the field-dependent reference states, ground state energies and spin-wave energy spectra for the different magnetic phases were determined. The results obtained in Part I are employed in this paper in calculating the dependence of the total and sublattice magnetizations and susceptibilities on the external magnetic field and temperature in the long-wavelength low-temperature approximation.

1. Introduction

In the first part [1] of this paper (hereafter denoted I) a two-sublattice Néel-type Heisenberg ferrimagnet with uniaxial nearest-neighbour exchange anisotropy and external magnetic field parallel or perpendicular to the anisotropy axis has been considered. Using the spin wave formalism the field-dependent reference states (spin deviation vacua), ground state energies and spin-wave energy spectra for the different magnetic phases have been derived in the free-particle approximation, and the zero temperature critical field strengths for the magnetic phase transitions have been determined. The results obtained in Part I will be utilized in the present paper to calculate the total and sublattice magnetizations, the magnetic susceptibilities and the specific heats in the different phases, and to discuss the dependence of these quantities on the temperature as well as on the external magnetic field strength. The influence of the spin-wave interactions in the lowest-approximation of the Holstein-Primakoff mapping will be investigated in a separate paper [2].

* Address: Instytut Niskich Temperatur i Badań Strukturalnych PAN, Wrocław, Pl. Katedralny 1, Poland.

2. *Thermodynamical calculations*

The Hamiltonian (I.9) of Part I has been brought to the diagonal form

$$H = H_0 + \Delta H_2 + \sum_k (E_k^{(1)} \alpha_k^+ \alpha_k^+ + E_k^{(2)} \beta_k^+ \beta_k^+) + \dots \tag{1}$$

where H_0, ΔH_2 and the spin wave energies E_k are given by Eqs (I.11), (I. 29), (I. 31) and are functions of the angles θ_1, θ_2. The latter have been chosen in such a way as to minimize H_0, in which case the linear term H_1 in (I. 9) vanishes, in accordance with the general proof given in [3].

For the Hamiltonian (1) the partition function Z has the form

$$Z = e^{-\beta(H_0+\Delta H_2)} \prod_{k,i} (1-e^{-\beta E_k{}^{(i)}})^{-1};$$

$$\beta = \frac{1}{kT}; \qquad i = 1, 2. \tag{2}$$

Hence, we have for the system's free energy

$$F = -\frac{1}{\beta} \ln Z = H_0 + \Delta H_2 + \frac{1}{\beta} \sum_{k,i} \ln (1-e^{-\beta E^{(i)}{}_k}) \tag{3}$$

and for the components of the sublattice magnetizations

$$m_{i\alpha}(T) = -\frac{1}{V} \frac{\partial F}{\partial H_{i\alpha}} = m_{i\alpha}(0) - \Delta m_{i\alpha}(T) \tag{4}$$

where $\alpha = x, z$ and

$$m_{i\alpha}(0) = -\frac{1}{V} \frac{\partial}{\partial H_{i\alpha}} (H_0 + \Delta H_2), \tag{5}$$

$$\Delta m_{i\alpha}(T) \equiv m_{i\alpha}(0) - m_{i\alpha}(T) = \frac{1}{V} \sum_{k,i} \frac{\partial E_k^{(i)}}{\partial H_{i\alpha}} \langle n_k^{(i)} \rangle, \tag{6}$$

$$\langle n_k^{(i)} \rangle = \frac{1}{e^{\beta E_k{}^{(i)}} - 1} = \sum_{n=1}^{\infty} e^{-n\beta E_k{}^{(i)}}. \tag{7}$$

The parameters $H_{i\alpha}$ represent the components of the homogeneous external magnetic field as introduced in the Hamiltonian (I. 1).

One must remember that the derivatives in (4), (5), (6) must be taken only with regard to the explicit dependence on the external magnetic field $H_{i\alpha}$, *i. e.*, not through the angles θ_i. Otherwise, one obtains erroneous formulae for the magnetization and susceptibility as, *e. g.*, in [4, 5].

The temperature-dependent part of the susceptibility tensor has the form

$$\Delta\chi_{\alpha\beta}^{(i)} = \frac{\partial}{\partial H_\beta} \Delta m_{i\alpha}(T) \tag{8}$$

and the spin specific heat is given by

$$C = \frac{dU}{dT} \tag{9}$$

where U is the internal energy

$$U = \sum_{k,i} E_k^{(i)} \langle n_k^{(i)} \rangle. \tag{10}$$

In contrast to (4), (5), (6) the derivatives in (8) have to be taken also with respect to the implicit dependence on H_α, *i. e.*, through the angles θ_i.

From Part I it follows that there are in each magnetic phase two branches of the spin wave energy spectrum: the "acoustical" branch E_k^- which has a minimum for $k = 0$, and the "optical" branch E_k^+ having a maximum at $k = 0$. Both branches are separated by an energy gap which is at least of the order of $\sim \bar{I}(S_2 - S_1)$. This is illustrated for the transversal-field case in Table I where the spin wave energies at the crucial points $\gamma_k = \pm 1$ and $\gamma_k = 0$ of the first Brillouin zone are given, for the zero-field and critical-field case and for $\mu_1 = \mu_2$.

TABLE I

	$\gamma_k = 0$	$\gamma_k = \pm 1$
$h_x = h_z = 0$	$E_k^+ = \bar{I}S_2Z$ $E_k^- = \bar{I}S_1Z$ $\Delta E = E_k^+ - E_k^- = \bar{I}Z(S_2 - S_1)$	$E_k^+ = \frac{1}{2}\bar{I}S_1[Z(\varkappa+1) + \sqrt{Z^2(\varkappa-1)^2 + 4\varkappa}]$ $E_k^- = \frac{1}{2}\bar{I}S_1[Z(\varkappa+1) - \sqrt{Z^2(\varkappa-1)^2 + 4\varkappa}]$ $\Delta E = \bar{I}S_1\sqrt{Z^2(\varkappa-1)^2 + 4\varkappa}$
$h_x = h_{xc};\ h_z = 0$	$E_k^+ = \bar{I}S_2(1 + h_{xc})$ $E_k^- = \bar{I}S_1(1 + \varkappa h_{xc})$ $\Delta E = \bar{I}(S_2 - S_1)$	$E_k^+ = IS_1$ $E_k^- = 0$ $\Delta E = E_k^+$

In the limit $\kappa = 1$ (*i. e.*, $S_1 = S_2$; ferromagnet) the energy gap vanishes and we have a simple energy spectrum of a ferromagnetic crystal with an elementary cell corresponding to that of the sublattice of the ferrimagnet ($\kappa \neq 1$).

Because our considerations are limited to low temperatures at which virtually only acoustical magnons exist, we may in the first approximation, certainly neglect the influence of the optical modes on the thermodynamical properties of the system. The latter will play a substantial role at higher temperatures.

As the spin wave energy spectrum (I. 31) is very complicated, we must make some approximations in simplifying it for further calculations. One possibility is to expand the spectrum with respect to the ratio $\frac{K}{I}$ combined with the long-wavelength approximation. The advantage of the latter approach has been demonstrated for ferromagnets in [7]. In our case the last method consists in the expansion of E_k^- with regard to the coefficient m_4 which satisfies the inequality $0 \leqslant m_4 \leqslant \frac{K}{I}$. Thus, we obtain

$$E_k^- = \tfrac{1}{2}\left[m_1+m_2-\sqrt{(m_1-m_2)^2+4m_3^2\gamma_k^2}\right]+o(m_4^2). \tag{11}$$

If we restrict the expansion (11) to the first term then we have $v_{nm} = 0$, $\Delta H_2 = 0$, and u_{nm} is given by Eq. (I. 36).

Expanding γ_k we obtain

$$\gamma_k = \frac{1}{z}\sum_{\delta} e^{ik\delta} = 1-\frac{1}{2z}\sum_{\delta}(k\delta)^2+\frac{1}{4z}\sum_{\delta}(k\delta)^4+\ldots \tag{12}$$

Terms with odd powers of $\boldsymbol{k\delta}$ vanish for lattices having inversion symmetry.

For a tetragonal lattice with lattice constants a and b formula (12) takes the form

$$\gamma_k = 1-z^{-1}[a^2(k_x^2+k_y^2)+b^2k_z^2]+\ldots \tag{13}$$

Hence, in the long-wavelength approximation the formula (11) may for a tetragonal crystal lattice be written as follows:

$$E_k^- = A+B[a^2(k_x^2+k_y^2)+b^2k_z^2]+\ldots \tag{14}$$

where

$$A = E_0^- = \tfrac{1}{2}\left[m_1+m_2-\sqrt{(m_1-m_2)^2+4m_3^2}\right], \tag{15}$$

$$B = \frac{2z^{-1}m_3^2}{\sqrt{(m_1-m_2)^2+4m_3^2}}.$$

Utilizing Eqs (3) and (14) and the standard approximation $\sum_k f(k) \to \frac{V}{(2\pi)^3}\int f(k)\, d^3k$, where V is the volume of the crystal, we get for the free-energy the approximate low-temperature formula (first approximation):

$$F = H_0-\frac{V}{(4\pi)^{3/2}a^2b}BZ_{5/2}(\beta A)\vartheta^{5/2} \tag{16}$$

where $Z_p(x) = \sum_{n=1}^{\infty} n^{-p}e^{-nx}$, $\vartheta = kT/B$.

In the above calculations the following formulas have been utilized:

$$\int_0^{\infty} x^n e^{-ax^2}dx = \frac{\Gamma[(n+1)/2]}{2a^{(n+1)/2}}; \quad a>0,\ n>-1; \tag{17}$$

$Z_p^{(n)}x = (-1)^n Z_{p-n}(x)$ where $\Gamma(z)$ is Euler's gamma function.

According to the definition (6) the temperature-dependent part of the magnetization is given by

$$\Delta m_{i\alpha}(T) = \frac{1}{(4\pi)^{3/2}a^2b}\{A'_{i\alpha}Z_{3/2}\vartheta^{3/2} + \tfrac{3}{2}BB'_{i\alpha}Z_{5/2}\vartheta^{5/2}\} \tag{18}$$

where $Z_p \equiv Z_p(\beta A)$; $i = 1, 2$; $\alpha = x, z$ and

$$(A'_{i\alpha}) = \left(\frac{\partial A}{\partial H_{i\alpha}}\right) \equiv \begin{pmatrix} A'_{1x}, & A'_{1z} \\ A'_{2x}, & A'_{2z} \end{pmatrix} =$$

$$= \begin{pmatrix} 1/2\mu_1[1-(m_1-m_2)L]\sin\theta_1, & 1/2\mu_1[1-(m_1-m_2)L]\cos\theta_1 \\ 1/2\mu_2[1+(m_1-m_2)L]\sin\theta_2, & 1/2\mu_2[1+(m_1-m_2)L]\cos\theta_2 \end{pmatrix}, \tag{19}$$

$$(B'_{i\alpha}) = \frac{\partial B}{\partial H_{i\alpha}} \equiv \begin{pmatrix} B'_{1x}, & B'_{1z} \\ B'_{2x}, & B'_{2z} \end{pmatrix} =$$

$$= \begin{pmatrix} -2z^{-1}\mu_1(m_1-m_2)L^3m_3^2\sin\theta_1, & -2z^{-1}\mu_1(m_1-m_2)L^3m_3^2\cos\theta_1 \\ 2z^{-1}\mu_2(m_1-m_2)L^3m_3^2\sin\theta_2, & 2z^{-1}\mu_2(m_1-m_2)L^3m_3^2\cos\theta_2 \end{pmatrix}, \tag{20}$$

$$L = [(m_1-m_2)^2+4m_3^2]^{-1/2}.$$

(The derivatives $A'_{i\alpha}$ and $B'_{i\alpha}$ are taken with respect only to the explicit dependence of m_1 and m_2 on $H_{i\alpha}$.)

For the temperature-dependent part of the susceptibility we have according to (8)

$$\begin{aligned}\Delta\chi^i_{\alpha\beta} = (4\pi)^{-3/2}(a^2b)^{-1}\{&-Z_{1/2}A'_{i\alpha}A'_\beta B^{-1}\vartheta^{1/2} + \\ &+Z_{3/2}[A''_{i\alpha\beta} - \tfrac{3}{2}B^{-1}(A'_{i\alpha}B'_\beta + A'_\beta B'_{i\alpha})]\vartheta^{3/2} + \\ &+\tfrac{3}{2}Z_{5/2}(B''_{i\alpha\beta} - \tfrac{5}{2}B'_{i\alpha}B'_\beta B^{-1})\vartheta^{5/2}\}\end{aligned} \tag{21}$$

where

$$A'_\beta = \frac{dA}{dH_\beta}, \quad B' = \frac{dB}{dH_\beta}, \quad A''_{i\alpha\beta} = \frac{d}{dH_\beta}(A'_{i\alpha}), \quad B''_{i\alpha\beta} = \frac{d}{dH_\beta}(B'_{i\alpha}),$$

$$H_\alpha \equiv H_{i\alpha}. \tag{22}$$

The spin specific heat (9) has in this approximation the form

$$\begin{aligned}C = \frac{k\cdot V}{(4\pi)^{3/2}a^2b}[&Z_{1/2}\cdot(A/B)^2\vartheta^{-1/2} + 3Z_{3/2}\cdot(A/B)\vartheta^{1/2} + \\ &+\tfrac{15}{4}Z_{5/2}\vartheta^{3/2}].\end{aligned} \tag{23}$$

The formulas (16, 18, 21, 23) are generally correct for a spin wave spectrum of the form (14).

3. *Longitudinal and transversal field case*

Our further discussion will be carried out according to the direction of the external magnetic field.

I. $h_x = 0$; $h_z \geqslant 0$ (longitudinal field)

We note that in this case the first term of the expansion (11) represents the exact form of the spin wave spectrum because $m_4 = 0$ if $h_x = 0$.

Upon specifying for $h_x = 0$ the coefficients $m_1, \ldots, m_3$ as defined by Eq. (I. 37), we obtain for (15), (19), (20), (22) the expressions given in Appendix A.

The assumption $\mu_1 = \mu_2 \equiv \mu$ (though $S_1 \neq S_2$) leads to remarkable simplifications of the formulas for the magnetizations and susceptibilities which in this case have the form:

$$\Delta m_z(T) = \Delta m_{1z}(T) + \Delta m_{2z}(T) = \frac{\mu}{(4\pi)^{3/2} a^2 b} \cdot Z_{3/2}(\beta A) \vartheta^{3/2}, \tag{24}$$

$$\Delta \chi_{zz}(T) = \Delta \chi_{zz}^{(1)} + \Delta \chi_{zz}^{(2)} = \frac{-\mu^2 B^{-1}}{(4\pi)^{3/2} a^2 b} \cdot Z_{1/2}(\beta A) \vartheta^{1/2}. \tag{25}$$

II. $h_x \neq 0$; $h_z = 0$ (transversal field)

a) $h_x < h_{xc}$ ("scissor phase")

If we assume $h_x \ll h_{xc}$, we may restrict the expansion (11) to the first term and the spin wave spectrum will appear in the form (14). For weak field strengths the formulas (I. 22a, b), (I. 23) may be written as follows:

$$\sin \theta_1 \cong (Z+\kappa) h_x / w; \quad \sin \theta_2 \cong (\kappa Z + 1) h_x / w,$$

$$\cos \theta_1 \cong \cos \theta_2 \cong 1 \tag{26}$$

and the coefficients $m_1, \ldots, m_4$ in Eq. (I. 15) have the form

$$m_1 \cong \bar{I} S_2 [Z + (Z+\kappa)(\kappa Z + 1) h_x^2 / w^2 + (Z+\kappa) h_x^2 / w],$$

$$m_2 \cong \bar{I} S_1 [Z + (Z+\kappa)(\kappa Z + 1) h_x^2 / w^2 + (\kappa Z + 1) \kappa h_x^2 / w],$$

$$m_3 \cong -\tfrac{1}{2} \bar{I} (S_1 S_2)^{1/2} [Z(Z+\kappa)(\kappa Z + 1) h_x^2 / w^2 + 2],$$

$$m_4 \cong 0. \tag{27}$$

Similarly to case I (longitudinal fied), by means of Eqs (27) we can determine all the coefficients in the formulas (18), (21), (23) for the magnetizations, susceptibilities and specific heat, respectively. The final expressions are rather lengthy and shall not be given here explicitly.

b) $h_x < h_{xc}$ ("paramagnetic" phase)

In this case the restriction to the first term of the expansion (11) requires the assumption $h_x \gg h_{xc}$. The coefficients in the thermodynamical quantities (18), (21), (23) can be calculated by taking for $m_1, \ldots, m_3$ the values

$$m_1 = \bar{I}S_2(1+h_x), \quad m_2 = \bar{I}S_1(1+\kappa h_x), \quad m_3 = -\tfrac{1}{2}\bar{I}(S_1S_2)^{1/2}(Z+1) \tag{28}$$

which follow from (I. 25) and (I. 15). The results are given in Appendix B.

c) $h_x = h_{xc}$

As it follows from (I. 45) the acoustical branch of the spin wave spectrum may be rewritten in the form

$$E_k^- = \frac{Q^{1/2}}{[P+(P^2-Q)^{1/2}]^{1/2}}. \tag{29}$$

In the case $h_x = h_{xc}$ we have

$$Q = (\bar{I}^2S_1S_2Z)^2(Z^2-\gamma_k^2)(1-\gamma_k^2), \tag{30}$$

$$P = \tfrac{1}{2}\bar{I}^2S_1^2[\tilde{S}^2(1+h_{xc})^2+(1+\kappa h_{xc})^2+\tilde{S}Z\gamma_k^2]. \tag{31}$$

In the long-wavelength limit Eq. (29) may be written as

$$E_k^- = Ak \tag{32}$$

where

$$A = \frac{a\bar{I}S_2Z[2(Z^2-1)]^{1/2}}{z^{1/2}[\tilde{S}^2(1+h_{xc})^2+(1+\kappa h_{xc})^2+\tilde{S}Z]^{1/2}}. \tag{33}$$

In deriving the approximate formula (32) we have assumed, for simplicity, $a = b$ (cubic lattice).

From (5) and (32) we obtain, after straightforward calculations,

$$\Delta m_{ix}(h_{xc}) = \frac{B_i}{A}\vartheta^2+o(\vartheta^4) \tag{34}$$

where

$$B_i = \frac{\mu_i}{24}\left\{m_i+\frac{(-1)^i m_i(m_1^2-m_2^2)-2m_1(m_3^2-m_4^2)-2m_2(m_3^2+m_4^2)}{[(m_1^2-m_2^2)^2+4m_3^2(m_1+m_2)^2-4m_4^2(m_1-m_2)^2]^{1/2}}\right\}, \tag{35}$$

$$m_1 = \bar{I}S_2(1+h_{xc}); \quad m_2 = \bar{I}S_1(1+\kappa h_{xc}),$$
$$m_3 = -\tfrac{1}{2}\bar{I}(S_1S_2)^{1/2}(Z+1); \quad m_4 = -\tfrac{1}{2}\bar{I}(S_1S_2)^{1/2}(Z-1). \tag{36}$$

4. *Concluding remarks*

In this paper we have applied the linear spin wave theory of Part I in determining the thermodynamical properties of a two-sublattice uniaxial ferrimagnet with nearest-neighbour exchange interactions and external magnetic field perpendicular (transversal) or parallel (longitudinal) to the anisotropy axis of the crystal. From our calculations it follows that for the longitudinal as well as transversal fields whose strength (in the latter case) is far below or above the critical value h_{xc}, the spin-wave energy spectrum is in the long-wavelength limit a parabolic function of the length of the wave vector k. This gives in the free energy the first term $\sim T^{5/2}$, similar to ferromagnets. In the formula for the magnetization, however, we have beyond the standard term $\sim T^{3/2}$ also the (uncomplete) term $\sim T^{5/2}$, in spite of the approximation (14). The latter term is a pure consequence of the different sublattice spins.

In a transversal field with strength $h_x \cong h_{xc}$ the spin wave energy appears to be linear in k. Thus, in this case the magnetization behaves like T^2.

In this paper we were working with a temperature-independent reference state (*i. e.*, the angles θ_i do not depend on the temperature), which results in the fact that, in the scissor phase the projection of the magnetization on the direction of the (transversal) magnetic field is at constant field strength a decreasing function of the temperature, cp. (4), (18). This result remains true in the ferromagnetic case $\kappa = 1$ and hence disagrees with that obtained in [4, 5]. A closer inspection shows that the result in [4, 5] is a consequence of incorrect differentiation of the free energy with regard to the magnetic field strength (cp. our comments of Eqs (4)–(6) in Section 2). Anyhow, both results disagree with that of the molecular field theory (MFA) [8] which (for ferromagnets) predicts this projection of the magnetization to be independent on the temperature. This problem will be analysed more closely in a separate paper [9] on introducing a temperature-dependent spin-wave vacuum.

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APPENDIX A

The coefficients (15), (19), (20), (22) are as follows

$$A = \frac{\bar{I}S_1}{2}\{Z(1+\tilde{S})+(\tilde{S}+\kappa)h_z-L\},$$

$$B = 2z^{-1}\tilde{S}IS_1L^{-1}; \quad L = \{[Z(\tilde{S}-1)+(\tilde{S}-\kappa)h_z]^2+4\tilde{S}\}^{1/2}. \tag{A.1}$$

$$A'_{ix} = B'_{ix} = 0,$$

$$A'_{1z} = \tfrac{1}{2}\mu_1\{1-[Z(\tilde{S}-1)+(\tilde{S}-\kappa)h_z]L^{-1}\}, \tag{A.2}$$

$$A'_{2z} = \tfrac{1}{2}\mu_2\{1+[Z(\tilde{S}-1)+(\tilde{S}-\kappa)h_z]L^{-1}\}.$$

$$B'_{1z} = -2z^{-1}\mu_1\tilde{S}[Z(\tilde{S}-1)+(\tilde{S}-\kappa)h_z]L^{-3},$$

$$B'_{2z} = -\frac{\mu_2}{\mu_1}\cdot B'_{1z}. \tag{A.3}$$

$$A''_{1zz} = \frac{-\mu_1^2(\tilde{S}-\kappa)2S}{IS_2L^3}, \quad A''_{2zz} = -\frac{\mu_2}{\mu_1}A''_{1zz},$$

$$B''_{1zz} = -\frac{2z^{-1}\mu_1^2(\tilde{S}-\kappa)\{4\tilde{S}-2[Z(\tilde{S}-1)+(\tilde{S}-\kappa)h_z]^2\}}{\bar{I}S_2L^5},$$

$$B''_{2zz} = -\frac{\mu_2}{\mu_1}B''_{1zz}. \tag{A.4}$$

$$A'_z = \tfrac{1}{2}\mu_1\tilde{S}^{-1}\{\tilde{S}+\kappa-L^{-1}[Z(\tilde{S}-1)+(\tilde{S}-\kappa)h_z](\tilde{S}-\kappa)\},$$

$$B'_z = B'_{1z}+B'_{2z}. \tag{A.5}$$

For $\mu_1 = \mu_2 = \mu$ the above formulas take the form

$$A = \frac{\bar{I}S_1}{2}\{Z(1+\tilde{S})+2\tilde{S}h_z-L\},$$

$$B = 2z^{-1}\tilde{S}\bar{I}S_1L^{-1}, \quad L = [Z^2(\tilde{S}-1)^2+4\tilde{S}]^{1/2}. \tag{A.6}$$

$$A'_{ix} = B'_{ix} = 0,$$

$$A'_{1z} = \tfrac{1}{2}\mu\{1-Z(\tilde{S}-1)L^{-1}\}, \quad A'_{2z} = \tfrac{1}{2}\mu\{1+Z(\tilde{S}-1)L^{-1}\}. \tag{A.7}$$

$$A''_{1zz} = A''_{2zz} = B''_{1zz} = B''_{2zz} = 0; \quad A'_z = \mu, \quad B'_z = 0. \tag{A.8}$$

$$B'_{1z} = -2z^{-1}\mu\tilde{S}Z(\tilde{S}-1)L^{-3}, \quad B'_{2z} = -B'_{1z}. \tag{A.9}$$

APPENDIX B

The coefficients in (18), (21), (23) are as follows

$$A = \tfrac{1}{2}\bar{I}S_1[(\tilde{S}+1)+(\tilde{S}+\kappa)h_x-L], \quad B = (2z)^{-1}\bar{I}\tilde{S}(Z+1)^2S_1L^{-1},$$

$$L = \{[\tilde{S}-1+(\tilde{S}-\kappa)h_x]^2+\tilde{S}(Z+1)^2\}^{1/2}. \tag{B.1}$$

$$A'_{1z} = A'_{2z} = 0, \quad A'_{1x} = \tfrac{1}{2}\mu_1\{1-[\tilde{S}-1+(\tilde{S}-\kappa)h_x]L^{-1}\},$$

$$A'_{2x} = \tfrac{1}{2}\mu_2\{1+[\tilde{S}-1+(\tilde{S}-\kappa)h_x]L^{-1}\},$$

$$B'_{1x} = -(2z)^{-1}\mu_1\tilde{S}(Z+1)^2[\tilde{S}-1+(\tilde{S}-\kappa)h_x]L^{-3}, \quad B'_{2x} = -\tilde{\mu}B'_{1x}. \tag{B.2}$$

$$A'_x = \tfrac{1}{2}\mu_1\{1+\tilde{\mu}-(1-\tilde{\mu})[\tilde{S}-1+(\tilde{S}-\kappa)h_x]L^{-1}\}, \quad \tilde{\mu} = \frac{\mu_2}{\mu_1},$$

$$B'_x = -\tfrac{1}{2}\mu_1z^{-1}(\tilde{S}-\kappa)(Z+1)^2[\tilde{S}-1+(\tilde{S}-\kappa)h_x]L^{-3},$$

$$A''_{1xx} = -\tfrac{1}{2}\frac{\mu_1^2}{\bar{I}S_2}(\tilde{S}-\kappa)\tilde{S}(Z+1)^2L^{-3}, \quad A''_{2xx} = -\tilde{\mu}A''_{1xx},$$

$$B''_{1xx} = -\frac{\mu_1^2}{2z}\frac{\tilde{S}(Z+1)^2(\tilde{S}-\kappa)}{IS_2}\{\tilde{S}(Z+1)^2-2[\tilde{S}-1+(\tilde{S}-\kappa)h_x]\}L^{-5},$$

$$B''_{2xx} = -\frac{\mu_2}{\mu_1}B''_{1xx}. \tag{B.3}$$

For $\mu_1 = \mu_2 = \mu$ we have

$$A = \tfrac{1}{2}\bar{I}S_1[\tilde{S}+1+2\tilde{S}h_x-L], \quad B = (2z)^{-1}\bar{I}\tilde{S}(Z+1)^2S_1L^{-1},$$

$$L = \{(\tilde{S}-1)^2+\tilde{S}(Z+1)^2\}^{1/2}. \tag{B.4}$$

$$A'_{1z} = A'_{2z} = 0, \quad A'_{1x} = \tfrac{1}{2}\mu\{1-(\tilde{S}-1)L^{-1}\}, \quad A'_{2x} = \tfrac{1}{2}\mu\{1+(\tilde{S}-1)L^{-1}\}, \tag{B.5}$$

$$B'_{1x} = -(2z)^{-1}\mu\tilde{S}(Z+1)^2(\tilde{S}-1)L^{-3}, \quad B'_{2x} = -B'_{1x},$$

$$A''_{1xx} = A''_{2xx} = B''_{1xx} = B''_{2xx} = 0. \tag{B.6}$$

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