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ON THE MECHANICAL RESPONSE OF A NON-UNIFORM PIEZOELECTRIC TRANSDUCER WITH ELASTIC COMPLIANCES HAVING DAMPING CHARACTERISTICS

By A. K. Pal

Department of Mathematics, Jadavpur University, Calcutta*

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The mechanical response of a non-uniform piezoelectric transducer with a damping characteristic of its elastic compliances is evaluated. It is seen that the response emitted is essentially similar to that for a transducer with compliances having uniform characteristics.

1. Introduction

In recent years there have been a large number of investigations on the responses, electrical and mechanical, of piezoelectric transducers in view of the use of these materials in ultrasonics. In support of this, the important works of Redwood [1], Mason [2], Filipczyński [3], Sinha [4], [5], Giri [6] and of others may be mentioned. But, in all these studies, the assigned inputs are always either a step function of force or of voltage and the discussions are mainly confined to the consideration of transducers having homogeneous material parameters. It is well known that crystals, particularly the piezoelectric ones often contain impurities which make them inhomogeneous. Olszak [7] has discussed the various types of inhomogeneities of material parameters of piezoelectric crystals. In this connection, the works of Chakravarti [8], Sinha [9] may also be noted. Again, in all these studies transducers having elastic compliances which remain entirely invariable with regard to time have been discussed.

In the present note, the elastic compliances are assumed to be partly constant and partly time-dependent. This assumption on the nature of elastic compliances is justified by the behaviour of electrets [12] and also by the similarity that the electrets have with piezoelectric materials [1]. The nature of non-uniformity considered in the present paper is in the sense used by Redwood and Mitchell [11]. The electric excitation considered is also in accordance with Redwood's assumption. Finally, the method of transform calculus has been used to facilitate the solution of the problem.

* Address: Department of Mathematics, Jadavpur University, Calcutta-32, India.

2. *Fundamental equations and boundary conditions*

Let us consider a piezoelectric-plate transducer, the thickness direction of which is taken as the x-axis. Let $x = 0$ and $x = X$ be the two extremities of the transducer. The electric excitation, according to Redwood's [11] assumption is taken to be

$$hD = h_0(1+Kx)\,H(t), \tag{1}$$

where h and D are the piezoelectric parameter and the electric flux density, h_0 and K are constants. $H(t)$ is the Heaviside unit function defined as follows

$$\begin{aligned} H(t) &= 0, \text{ when } t < 0 \\ &= 1, \text{ when } t > 0. \end{aligned} \tag{2}$$

The other end of the transducer is assumed to be held rigidly at the back. The fundamental equations of the problem are

$$T = c\left(\frac{\partial \xi}{\partial x}\right) - hD \tag{3}$$

$$\varrho \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial T}{\partial x}. \tag{4}$$

Here in accordance with the assumption, the elastic compliances are taken as

$$C = C_1 + C_2 \frac{\partial}{\partial t} \tag{5}$$

where C_1 and C_2 are constants. T is the mechanical stress, ϱ the density, ξ the displacement *etc.*

Combining equations (3), (4) and (5) we get,

$$\left(C_1 + C_2 \frac{\partial}{\partial t}\right)\frac{\partial^2 \xi}{\partial x^2} - \varrho \frac{\partial^2 \xi}{\partial t^2} = h_0 K H(t). \tag{6}$$

Taking Laplace transform, we get

$$\frac{d^2\bar{\xi}}{dx^2} - \left(\frac{\varrho p^2}{C_1 + C_2 p}\right)\bar{\xi} = \frac{h_0 K}{p(C_1 + C_2 p)}. \tag{7}$$

Its solution

$$\bar{\xi} = A \exp\left(-p\sqrt{\frac{\varrho}{C_1 + C_2 p}}\,x\right) + B \exp\left(+p\sqrt{\frac{\varrho}{C_1 + C_2 p}}\,x\right) - \frac{h_0 K}{\varrho p^3} \tag{8}$$

where A and B are functions of the parameter p.

The equation of force for the non-piezoelectric transducer as in Redwood [1] can be taken as

$$\bar{F} = pZc\left\{-A\exp\left(\frac{-px}{v}\right) + B\exp\left(\frac{+px}{v}\right)\right\} \tag{9}$$

where

$$v^2 = \frac{c}{\varrho} \tag{10}$$

and

$$Z_c = \varrho v S. \tag{11}$$

The corresponding equation for the piezoelectric transducer is

$$\bar{F} + \frac{Sh_0(1+Kx)}{p} = Sp\sqrt{\varrho}\sqrt{C_1+C_2p}\left\{-A\exp\left(-p\sqrt{\frac{\varrho}{C_1+C_2p}}\,x\right) + \right.$$

$$\left. + B\exp\left(+p\sqrt{\frac{\varrho}{C_1+C_2p}}\,x\right)\right\}. \tag{12}$$

The most general problem of this type can be thought of as consisting of a transducer of impedance Z_c situated between the materials of impedances Z_1 and Z_2. Then the conditions of continuity of the force and displacement at $x = 0$ and $x = X$ provide the equations which when solved will yield the values of the constants like A, B, *etc.*

These conditions are

(a) at $x = 0$,

$$(\bar{F}_1)_0 = (\bar{F})_0$$

$$(\bar{\xi}_1)_0 = (\bar{\xi})_0 \tag{13}$$

(b) at $x = X$,

$$(\bar{F}_2)_X = (\bar{F})_X$$

$$(\bar{\xi}_2)_X = (\bar{\xi})_X \tag{14}$$

where the suffixes 1 and 2 represent the relevant quantities at $x = 0$ and $x = X$ respectively.

3. *Solution of the problem*

To simplify the calculations, we assume the transducer to be rigidly backed at the end $x = X$. So we can take

$$A_2 = B_2 = A_1 = 0.$$

Hence, from the continuity conditions of force and displacement we have the following set of equations

$$A\exp\left(-p\sqrt{\frac{\varrho}{C_1+C_2p}}\,x\right) + B\exp\left(+p\sqrt{\frac{\varrho}{C_1+C_2p}}\,x\right) - \frac{h_0K}{\varrho p^3} = 0 \tag{15}$$

$$Sp\sqrt{\varrho}\sqrt{C_1+C_2p}\,(-A+B)-\frac{Sh_0}{p}=pZ_1B_1 \tag{16}$$

$$A+B-\frac{h_0K}{\varrho p^3}=B_1. \tag{17}$$

Solving these equations we obtain

$$A=\frac{-\left(\frac{Sh_0}{p^2}-\frac{h_0KZ_1}{\varrho p^3}\right)e^{+p\sqrt{\frac{\varrho}{C_1+C_2p}}x}-\frac{h_0K}{\varphi p^3}\{Z_1-S\sqrt{\varphi}\sqrt{C_1+C_2p}\}}{e^{+p\sqrt{\frac{\varrho}{C_1+C_2p}}x}\{Z_1+S\sqrt{\varrho}\sqrt{C_1+C_2p}\}-e^{-p\sqrt{\frac{\varrho}{C_1+C_2p}}x}\{Z_1-S\sqrt{\varrho}\sqrt{C_1+C_2p}\}} \tag{18}$$

$$B=\frac{\frac{h_0K}{\varrho p^3}\{Z_1+S\sqrt{\varrho}\sqrt{C_1+C_2p}\}+\left(\frac{Sh_0}{p^2}-\frac{h_0KZ_1}{\varrho p^3}\right)e^{-p\sqrt{\frac{\varrho}{C_1+C_2p}}x}}{e^{+p\sqrt{\frac{\varrho}{C_1+C_2p}}x}\{Z_1+S\sqrt{\varrho}\sqrt{C_1+C_2p}\}-e^{-p\sqrt{\frac{\varrho}{C_1+C_2p}}x}\{Z_1-S\sqrt{\varrho}\sqrt{C_1+C_2p}\}} \tag{19}$$

Substituting these values of A and B in the equation for $(\bar{\xi})_0$ and simplifying we get

$$(\bar{\xi})_0=\frac{-\{\varrho Sh_0p+h_0KS\sqrt{\varrho}\sqrt{C_1+C_2p}\}}{\varrho p^3\{Z_1+S\sqrt{\varrho}\sqrt{C_1+C_2p}\}}= \tag{20}$$

$$=\frac{-\left\{\varrho Sh_0p+h_0KS\sqrt{\varrho}\sqrt{C_2}\sqrt{p+\frac{C_1}{C_2}}\right\}}{\varrho p^3\left\{Z_1+S\sqrt{\varrho}\sqrt{C_2}\sqrt{p+\frac{C_1}{C_2}}\right\}}. \tag{21}$$

Let us suppose

$$\frac{C_1}{C_2}=a$$

$$(\bar{\xi})_0=\frac{-Sh_0}{p^2\{Z_1+S\sqrt{\varrho}\sqrt{C_2}\sqrt{p+a}\}}-\frac{h_0KS\sqrt{\varrho}\sqrt{C_2}\sqrt{p+a}}{\varrho p^3\{Z_1+S\sqrt{\varrho}\sqrt{C_2}\sqrt{p+a}\}}=$$

$$=\frac{-h_0K}{\varrho p^3}-\frac{Sh_0}{p^2\{Z_1+S\sqrt{\varrho}\sqrt{C_2}\sqrt{p+a}\}}+\frac{h_0KZ_1}{\varrho p^3\{Z_1+S\sqrt{\varrho}\sqrt{C_2}\sqrt{p+a}\}}=$$

$$=-\frac{h_0K}{\varrho p^3}-\frac{Sh_0}{p^2S\sqrt{\varrho}\sqrt{C_2}\,(b+\sqrt{p+a})}+\frac{h_0KZ_1}{\varrho p^3S\sqrt{\varrho}\sqrt{C_2}\,(b+\sqrt{p+a})} \tag{22}$$

where

$$b=\frac{Z_1}{S\sqrt{\varrho}\sqrt{C_2}}. \tag{23}$$

Applying the Vanderpol and Bremmer technique

$$\xi(0,t)e^{at} = -\frac{h_0 K}{\varrho(p-a)^3} - \frac{Sh_0}{S\sqrt{\varrho}\sqrt{C_2}\,(p-a)^2\{\sqrt{p+b}\}} + \\ + \frac{h_0 KZ_1}{\varrho S\sqrt{\varrho}\sqrt{C_2}\,(p-a)^3\{\sqrt{p+b}\}}. \tag{24}$$

Taking the inverse transform we obtain

$$(\xi)_0 \quad \left\{\frac{h_0}{\sqrt{\varrho}\sqrt{C_2}}\left[\frac{1}{\sqrt{a}\,(a-b^2)^2} - \frac{t}{\sqrt{a}\,(a-b^2)}\right] + \frac{h_0 KZ_1}{\varrho S\sqrt{\varrho}\sqrt{C_2}}\left[\frac{1}{\sqrt{a}\,(a-b^2)^3} - \right.\right.$$

$$\left. - \frac{t}{\sqrt{a}\,(a-b^2)^2} + \frac{t^2}{4\sqrt{a}\,(a-b^2)^2}\right] + \frac{\sqrt{\pi}}{2a^{3/2}}\left[\frac{h_0}{\sqrt{\varrho}\sqrt{C_2}\sqrt{\pi}}\cdot\frac{1}{(a-b^2)^2} + \right.$$

$$\left. + \frac{h_0 KZ_1}{\varrho S\sqrt{\varrho}\sqrt{C_2}\sqrt{\pi}}\,\frac{1}{(a-b^2)^2} - \frac{h_0 KZ_1}{\varrho S\sqrt{\varrho}\sqrt{C_2}\sqrt{\pi}}\cdot\frac{t}{(a-b^2)^2}\right] +$$

$$\left. + \frac{3h_0 KZ_1}{8\varrho S\sqrt{\varrho}\sqrt{C_2}\,a^{5/2}(a-b^2)^2}\right\} Erf(\sqrt{at}) -$$

$$- \left\{\frac{h_0}{\sqrt{\varrho}\sqrt{C_2}}\cdot\frac{e^{(b^2-a)t}}{b(a-b^2)^2} + \frac{h_0 KZ_1}{\varrho S\sqrt{\varrho}\sqrt{C_2}}\,\frac{e^{(b^2-a)t}}{b(a-b^2)^3}\right\} Erf(b\sqrt{t}) +$$

$$+ \frac{h_0 b}{\sqrt{\varrho}\sqrt{C_2}}\left\{\frac{t}{(a-b^2)} - \frac{1}{(a-b^2)^2} + \frac{e^{(b^2-a)t}}{(a-b^2)}\right\} + \frac{h_0 KZ_1 b}{\varrho S\sqrt{\varrho}\sqrt{C_2}}\left\{\frac{e^{(b^2-a)t}}{(a-b^2)^3} - \right.$$

$$\left. - \frac{t^2}{2(a-b^2)} + \frac{t}{(a-b^2)^2} - \frac{1}{(a-b^2)^3}\right\} - \frac{h_0 K t^2}{2\varrho} -$$

$$- \frac{1}{a}e^{-at}t^{1/2}\left\{\frac{h_0}{\sqrt{\varrho}\sqrt{C_2}\sqrt{\pi}}\cdot\frac{1}{(a-b^2)^2} + \frac{h_0 KZ_1}{\varrho S\sqrt{\varrho}\sqrt{C_2}\sqrt{\pi}}\cdot\frac{1}{(a-b^2)^2} - \right.$$

$$\left. - \frac{h_0 KZ_1}{\varrho S\sqrt{\varrho}\sqrt{C_2}\sqrt{\pi}}\cdot\frac{t}{(a-b^2)^2}\right\} - \left\{\frac{3}{2a^2}e^{-at}\cdot t^{1/2} + \frac{1}{a}e^{-at}\cdot t^{3/2}\right\} \times$$

$$\times \frac{h_0 KZ_1}{2\varrho S\sqrt{\varrho}\sqrt{C_2}\sqrt{\pi}\,(a-b^2)^2}. \tag{25}$$

This gives the mechanical response emitted by a non-uniform transducer, where the elastic compliances have damping characteristics. The response reduces to zero at $t = 0$, in accordance with the initial conditions of the problem. If we put $C_2 = 0$ in the equation

(20) the results obtained agree with those, obtained for constant elastic compliances by the present author in one of his papers. It is found that some additional time-dependent terms appear due to the damping nature of the compliances.

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