Vol. A42 (1972) ACTA PHYSICA POLONICA No 5

ONE-DIMENSIONAL MODEL OF THE REARRANGEMENT AND DISSOCIATION PROCESSES

By K. Lipszyc

Institute of Physical Chemistry, Polish Academy of Sciences, Warsaw*

(Received February 12, 1972; Revised paper received April 8, 1972)

The two-dimensional diffraction problem is shown to be mathematically identical with a certain quantum mechanical model of collision of three collinear particles. An asymptotic expression of the solution of the wave equation was used to discuss the probabilities of the elastic, rearrangement and dissociation processes. The model of three particles with equal masses and equal potentials is discussed as a particular case of the solution under consideration.

1. Introduction

The one-dimensional models of collision processes have been studied extensively [1] – [6]. It is well known that calculations of the transition amplitudes for the processes above the dissociation threshold still present considerable difficulties.

We came across a solution of a scattering problem in acoustics mathematically very similar to a certain quantum mechanical scattering problem of three collinear particles. This quantum mechanical model, although drastically simplified, exhibits the essential features of the three-body problem: it describes the rearrangement process and the dissociation of a pair of bound particles under the influence of the third one.

First, we shall discuss the two-body subsystems needed for the solution of the three-body problem. Then, we show that in our case the Schrödinger equation for three particles is identical with the wave equation in acoustics and we shall interpret the solution in the "particle language".

2. The two-body system

The two-body potential we shall discuss is of zero range and its effect on the wave function can be incorporated into the Schrödinger equation

$$-\frac{d^2}{dx^2}\Psi(x) = \hat{E}\Psi(x) \tag{1}$$

* Address: Instytut Chemii Fizycznej PAN, Warszawa 42, Kasprzaka 44/52, Poland.

as a boundary condition

$$\frac{d}{dx}\Psi(x)+\alpha\Psi(x) = 0 \qquad \text{for } x = 0 \tag{2}$$

where x is the coordinate of the relative position of paricles i and j

$$x = x_i - x_j = (r_i - r_j)\left[\frac{2m_i m_j}{m_i+m_j}\right]^{\frac{1}{2}}$$

$$-\infty < r_i, r_j < \infty \tag{3}$$

and α is a real number. We put $\hbar = 1$.

The equation (1) is written in the centre-of-mass systems of the two particles.

The solutions of the equation (1) satisfying the condition (2) are different from zero on one semi-axis x only (the potential is impenetrable). Proceeding further, when constructing the three-body system, we shall consider the class of two-body wave functions different from zero on the semi-axis $x < 0$

$$\Psi(x) = e^{i\sqrt{\hat{E}}x} - \frac{\alpha+i\sqrt{\hat{E}}}{\alpha-i\sqrt{\hat{E}}}e^{-i\sqrt{\hat{E}}x} \qquad \text{for } x < 0.$$

$$\Psi(x) = 0 \qquad \text{for } x > 0.$$

If $\alpha < 0$, there exists one bound state at the energy: $E = -\alpha^2 = -E_b$ for which (4)

$$\Psi_b(x) = (-2\alpha)^{\frac{1}{2}}e^{-\alpha x} \qquad \text{for } x < 0$$

$$\Psi_b(x) = 0 \qquad \text{for } x > 0. \tag{5}$$

The two-body t-matrix for the potential under consideration can be calculated applying the method described in Ref. [7]. It takes the one-component separable form

$$\langle p'|t(z)|p\rangle = (p'+i\alpha)\frac{2i\sqrt{z}}{z+\alpha^2}(-p+i\alpha)$$

$$-\infty < p, p' < \infty. \tag{6}$$

Applying the limiting procedure $(\alpha) \to \infty$ to the expression (6) we obtain the expression for the hard-rod t-matrix [7]

$$\langle p'|t(z)|p\rangle \to -2i\sqrt{z}. \tag{7}$$

The two-body t-matrix off the energy shell, *i. e.* for complex z, is involved in the calculations of the three-body T-matrix [8].

The zero-range potential may be considered as a limiting case of a very narrow and deep square well potential with adjacent impenetrable wall, *i. e.* of the potential

$$v(x) = -v_0 \qquad \text{for } -\delta < x < 0$$

$$v(x) = 0 \qquad \text{for } -\infty < x < -\delta$$

$$v(x) = \infty \qquad \text{for } x = 0. \tag{8}$$

The condition for existence of exactly one bound state is

$$\frac{\pi}{2} < \sqrt{v_0}\,\delta < \frac{3\pi}{2} \tag{9}$$

and the binding energy is given by the equation describing the condition of smoothness of the wave function at $x = -\delta$

$$\frac{\Psi'(x)}{\Psi(x)} = -\sqrt{v_0 - E_b}\,\frac{\cos(\sqrt{v_0 - E_b}\,\delta)}{\sin(\sqrt{v_0 - E_b}\,\delta)} = \sqrt{E_b} \quad \text{for } x = -\delta. \tag{10}$$

If we apply the following limiting procedure

$$v_0\delta^2 \underset{\delta\to 0}{\longrightarrow} \frac{\pi^2}{4} - 2\alpha\delta \quad E_b < \infty \tag{11}$$

to the left-hand side of equation (10), assuming that E_b remains finite, we obtain that the logarithmic derivative of $\Psi(x)$ at $x = -\delta$ tends to a constant value of $-\alpha$, thus verifying in the limit the condition (2).

The boundary condition of the type (2) has been often used in nuclear physics [9], [10]. It is imposed on the function $u(r) = rR(r)$, where $R(r)$ is the radial part of the three--dimensional wave function, and it acts as a central zero-range "delta function" potential. The essential difference between the one-dimensional potential defined by (2) and the radial "delta function" potential is that the former contains the "hard core" component, whereas the latter does not.

When the absolute value of α tends to infinity the condition (2) takes the form

$$\Psi(x) = 0 \quad \text{for } x = 0 \tag{2a}$$

which describes the potential of a hard rod in one dimension. In the three-body model we shall also discuss the case when one of the two-body potentials is of this kind. This will be the model in which the scattering on a bound state leads to either elastic scattering or dissociation and no rearrangement takes place.

3. The three-particle system

The position and momentum coordinates used in the three-body system are defined as follows

$$\begin{aligned} s_1 &= (r_2 - r_3)\left[\frac{2m_2m_3}{m_2+m_3}\right]^{\frac{1}{2}} \\ t_1 &= \left(\frac{m_2r_2+m_3r_3}{m_2+m_3} - r_1\right)\left[\frac{2m_1(m_2+m_3)}{m_1+m_2+m_3}\right]^{\frac{1}{2}} \end{aligned} \tag{12}$$

$$\begin{aligned} p_1 &= \frac{m_3k_2 - m_2k_3}{[2m_2m_3(m_2+m_3)]^{\frac{1}{2}}} \\ q_1 &= \frac{m_1(k_2+k_3) - (m_2+m_3)k_1}{[2m_1(m_2+m_3)(m_1+m_2+m_3)]^{\frac{1}{2}}} \end{aligned} \tag{12a}$$

where r_i and k_i are the position and momentum coordinates respectively in the three--body centre-of-mass system. We shall also use the position coordinates s_3 and t_3 as well as the momentum coordinates p_3 and q_3. The system (s_1, t_1) is related to that of (s_3, t_3) by the transformation

$$s_1 = -bs_3 + at_3$$
$$t_1 = -as_3 - bt_3 \tag{13}$$

where

$$a = \left[\frac{m_2(m_1+m_2+m_3)}{(m_1+m_2)(m_2+m_3)}\right]^{\frac{1}{2}}$$

$$b = \left[\frac{m_1 m_3}{(m_1+m_2)(m_2+m_3)}\right]^{\frac{1}{2}}.$$

The matrix of transformation for the coordinates (p, q) is the same as in (13).

We choose the ordering of particles (1 2 3), *i. e.* $r_3 > r_2 > r_1$ and, the applied potentials being impenetrable, the original order cannot ever change. Hence, we always have $s_1 < 0$ and $s_3 < 0$.

The Schrödinger equation for the three particles in the centre-of-mass system takes the form

$$\left[-\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial t^2}\right] \Psi(s, t) = E\Psi(s, t) \tag{14}$$

with the boundary conditions describing the action of potentials between the pairs (1, 2) and (2, 3)

$$\frac{\partial \Psi}{\partial s_3} + \alpha_3 \Psi(s_3, t_3) = 0 \quad \text{for } s_3 = 0 \tag{15a}$$

$$\frac{\partial \Psi}{\partial s_1} + \alpha_1 \Psi(s_1, t_1) = 0 \quad \text{for } s_1 = 0. \tag{15b}$$

The fact that we may incorporate the two-body potentials into the three-particle equation in the form (15) is the result of the zero-range approximation for the two-particle forces. Suppose we consider the Schrödinger equation

$$[-\Delta + V_1(s_1) + V_3(s_3)]\, \Psi(s, t) = E\Psi(s, t) \tag{16}$$

where V_1 and V_3 are the square well potentials of the type (8), as shown in Fig. 1.

The solution of Eq. (16) may be formally expressed by the equation

$$\Psi(s, t) = \int ds'dt' G_{\text{hard rod}}(s, t; s', t'; E) \times$$
$$\times (V_1' + V_3')\, \Psi(s', t') \tag{17}$$

where V_1' and V_3' represent the attractive components of V_1 and V_3 and $G_{\text{hard rod}}$ is the Green function equal to zero at the lines $s_1 = 0$ and $s_3 = 0$. If in the initial state three particles move freely, the plane wave term should be added to the right-hand side of Eq. (17). We see that $\Psi(s, t)$ is entirely determined by its values in the region where $V_1' + V_3'$

is different from zero. In the region where only one potential is different from zero the wave function should behave as follows

$$\Psi(s, t) \cong \int dp\, [C_3(p) \sin(\sqrt{E+v_{03}-p^2}\, s_3)\, e^{ipt_3}]$$

or respectively

$$\Psi(s, t) \cong \int dp\, [C_1(p) \sin(\sqrt{E+v_{01}-p^2}\, s_1)\, e^{ipt_1}]. \tag{18}$$

If the radii δ_1 and δ_3 of the potentials V_1 and V_3 tend to zero, the region where both V_1 and V_3 are different from zero contracts to a point. If the limiting procedures are of

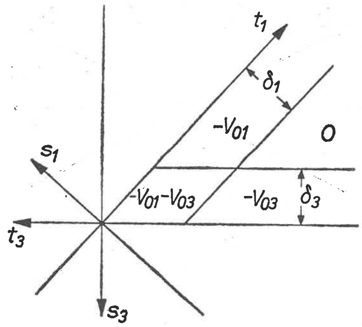

Fig. 1. The potentials $V_1 + V_3$ as the function of the three-particle coordinates

the type described by (11) the logarithmic derivatives of $\Psi(s, t)$ with respect to s_1 and s_3 at the lines $s_1 = -\delta_1$ and $s_3 = -\delta_3$ respectively, tend to constant values α_1 and α_3 provided p remains finite. Neglecting those parts of the integrand in (18) for which p may compete with v_0 (tending to infinity in the limiting procedure) seems justified because of its strongly oscillatory character for large p. We see that in the zero-range approximation the behaviour of the wave function inside the region of action of potentials is described by the values of the logarithmic derivatives at the lines $s_3 = 0$ and $s_1 = 0$, these values being the same as in the corresponding two-body problems.

Instead of the condition (15b) we shall also use the condition

$$\Psi(s_1, t_1) = 0 \quad \text{for } s_1 = 0 \tag{15b'}$$

i. e. we shall assume a hard-core interaction between the pair (2, 3).

The potential V_2 between the pair (1, 3) is put equal to zero.

Eqs (14)–(15) determine the boundary value problem for the two-dimensional wave equation. It is convenient to introduce the polar coordinates (r, φ)

$$s_3 = -r \sin(\Phi + \varphi)$$

$$t_3 = -r \cos(\Phi + \varphi)$$

$$s_1 = -r \sin(\Phi - \varphi)$$

$$t_1 = r \cos(\Phi - \varphi) \tag{19}$$

where

$$\mathrm{tg}\, 2\Phi = \left[\frac{m_2(m_1+m_2+m_3)}{m_1 m_3}\right]^{\frac{1}{2}} \tag{20}$$

$\sin 2\Phi = a$; $\cos 2\Phi = b$ (see (13)).

The coordinate system is shown in Fig. 2.

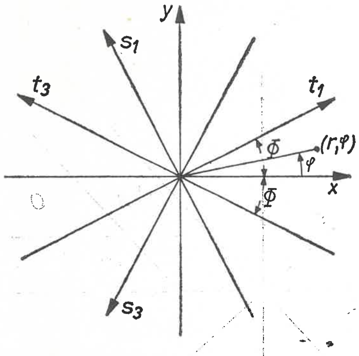

Fig. 2. The three-body coordinate system. The region where the order of particles is (1 2 3) is limited by the lines $\varphi = \Phi$ and $\varphi = -\Phi$

Using the polar coordinates we can write the Eqs (14) and (15) in the form

$$-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r}\frac{\partial^2}{\partial\varphi^2}\right]\Psi(r,\varphi) = E\Psi(r,\varphi) \tag{21}$$

with the boundary conditions

$$-\frac{1}{r}\frac{\partial}{\partial\varphi}\Psi(r,\varphi)+\alpha_3\Psi(r,\varphi) = 0 \quad \text{for } \varphi = -\Phi \tag{22a}$$

$$\frac{1}{r}\frac{\partial}{\partial\varphi}\Psi(r,\varphi)+\alpha_1\Psi(r,\varphi) = 0 \quad \text{for } \varphi = \Phi \tag{22b}$$

or

$$\Psi(r,\varphi) = 0 \quad \text{for } \varphi = \Phi. \tag{22b'}$$

This equation was solved by Maluzhinetz [11] when discussing the problem of scattering of plane and surface waves from the wedge with given face impedances. The boundary conditions (22) in the acoustics problem read as follows

$$\frac{1}{r}\frac{\partial}{\partial\varphi}\Psi(r,\varphi)\mp ik\sin\theta_\pm = 0 \quad \text{for } \varphi = \pm\Phi \tag{23}$$

where $k^2 = E$, and $\sin\theta_\pm$ are determined by the properties of the boundaries at $\varphi = \pm\Phi$.

In order to obtain the same notation as in Maluzhinetz work we put

$$\alpha_1 = -ik \sin \theta_+ : \quad \alpha_3 = -ik \sin \theta_- \tag{24}$$

keeping in mind that the values θ_- and θ_+ in the three-particle problem depend on both the total energy k^2 and the two-body constant α_3 or k^2 and α_1 respectively

$$\theta_- = \operatorname{arctg} \frac{i\alpha_3}{\sqrt{k^2+\alpha_3^2}} = \frac{1}{2i} \ln \frac{\sqrt{k^2+\alpha_3^2} - \alpha_3}{\sqrt{k^2+\alpha_3^2} + \alpha_3}$$

$$\theta_+ = \operatorname{arctg} \frac{i\alpha_1}{\sqrt{k^2+\alpha_1^2}} = \frac{1}{2i} \ln \frac{\sqrt{k^2+\alpha_1^2} - \alpha_1}{\sqrt{k^2+\alpha_1^2} + \alpha_1}. \tag{25}$$

In the quantum mechanical problem we are interested in the case

$$\begin{gathered} \operatorname{Im} \alpha_1 = 0 = \operatorname{Im} \alpha_3 \\ \operatorname{Re} \alpha_1 < 0, \ \operatorname{Re} \alpha_3 < 0. \end{gathered} \tag{26}$$

Using (24) we see that (26) is equivalent to the condition

$$\begin{gathered} \operatorname{Re} \theta_\pm = 0 \\ \operatorname{Im} \theta_\pm < 0 \quad \text{if } k > 0. \end{gathered} \tag{27}$$

In acoustics the condition (27) corresponds to the case of purely elastic impedances.

The meaning of equalities (24) becomes clearer if we introduce polar coordinates for the momenta p and q

$$\begin{aligned} p_3 &= -k \sin(\Phi + \varphi') \\ q_3 &= -k \cos(\Phi + \varphi') \\ p_1 &= -k \sin(\Phi - \varphi') \\ q_1 &= +k \cos(\Phi - \varphi'). \end{aligned} \tag{28}$$

The plane wave travelling from the reaction centre can be expressed in terms of these co-ordinates as follows

$$\begin{aligned} \exp\{is_1 p_1 + it_1 q_1\} &= \exp\{is_3 p_3 + it_3 q_3\} = \\ &= \exp\{ikr \cos(\varphi - \varphi')\} \end{aligned} \tag{29}$$

and the state in which two particles are bound and the third one leaves freely the reaction centre, as follows

$$\begin{aligned} \exp\{-\alpha_3 s_3 + it_3 q_3\} &= \exp\{-ik \sin \theta_- r \sin(\Phi + \varphi) + \\ &+ ikr \cos \theta_- \cos(\Phi + \varphi)\} = \\ &= \exp\{ikr \cos(\varphi + \Phi + \theta_-)\} \end{aligned}$$

or

$$\begin{aligned} \exp\{-\alpha_1 s_1 + it_1 q_1\} &= \exp\{ik \sin \theta_+ r \sin(\Phi - \varphi) + \\ &+ ikr \cos \theta_+ \cos(\Phi - \varphi)\} = \\ &= \exp\{ikr \cos(\varphi - \Phi - \theta_+)\}. \end{aligned} \tag{30}$$

We see that the "channel" wave functions describing the situation in which two particles are bound and the third one departs freely may be represented by a "plane wave" moving in a complex direction $\varphi' = -\Phi-\theta_-$ (channel 3) or in the direction $\varphi' = \Phi+\theta_+$ (channel 1).

The incident wave is written in the form: $e^{-ikr\cos(\varphi-\varphi')}$, where: φ' is real or $\varphi' = \Phi-\theta_+$ or $\varphi' = -\Phi+\theta_-$.

Following the papers [11c] and [11d] we shall discuss the solution of the wave equation with $E = k^2 > 0$, which corresponds to the situation above the three-body threshold.

The solution $\Psi(r, \varphi)$ is written in the form of the generalized Fourier transform

$$\Psi(r, \varphi) = \frac{1}{2\pi i}\int_{\gamma} e^{-ikr\cos z} s(z+\varphi)dz \tag{31}$$

where γ is the contour in the complex z plane known as the Sommerfeld's contour. It is shown for the case of real k in the Fig. 3.

The function $\Psi(r, \varphi)$ is uniquelly determined by the boundary conditions at the lines $\varphi = \Phi$ and $\varphi = -\Phi$, and by the radiation condition at large distances from the origin,

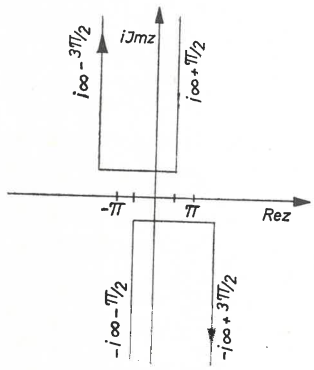

Fig. 3. Sommerfeld's contour

as well as by the boundness condition at $r = 0$. Maluzhinetz has shown that the transform $s(z)$ of the Sommerfeld's integral (31) can be written in the form of the finite combination of the special functions $M_\Phi(z)$ with different complex arguments z, and of trigonometric functions

$$s(z) = \frac{\pi}{2\Phi}\cos\frac{\pi\varphi_0}{2\Phi}\left(\sin\frac{\pi z}{2\Phi}-\sin\frac{\pi\varphi_0}{2\Phi}\right)^{-1} F(z)\,[F(\varphi_0)]^{-1} \tag{32}$$

where

$$F(z) = M_\Phi\left(z+\Phi+\frac{\pi}{2}-\theta_+\right)M_\Phi\left(z-\Phi-\frac{\pi}{2}+\theta_-\right)\times$$
$$\times M_\Phi\left(z+\Phi-\frac{\pi}{2}+\theta_+\right)M_\Phi\left(z-\Phi+\frac{\pi}{2}-\theta_-\right) \tag{33}$$

φ_0 is the direction of the incident wave.

The special function $M_\Phi(z)$ is defined by the equations

$$M(z) = \prod_{n=1}^{\infty}\prod_{m=1}^{\infty}\left[1-\left(\frac{z}{2\Phi(2n-1)+\frac{\pi}{2}(2m-1)}\right)\right]^{(-1)^{m+1}} =$$

$$= \exp\left[\frac{1}{8\Phi}\int_0^z d\mu \int_{-i\infty}^{i\infty} \operatorname{tg}\frac{\pi v}{4\Phi}\,\frac{dv}{\cos(v-\mu)}\right]. \tag{34}$$

The function $M_\Phi(z)$ is useful in solving several diffraction problems in wedge shaped regions [11]–[13]. Some of its properties are given in the Appendix.

Putting the expression (32) into (31) we obtain the closed form of the solution of the wave equation and hence also of the three-particle problem. Because we are interested in calculating the transition amplitudes we have to study the asymptotic behaviour of the solution at large distances from the origin. For this purpose we make use of the asymptotic expression given by Maluzhinetz [11c]

$$\Psi(r,\varphi) \cong f(\varphi,\varphi_0)\,(2\pi kr)^{-\frac{1}{2}} e^{i\left(kr+\frac{\pi}{4}\right)} +$$

$$+C_+ e^{ikr\cos(\varphi-\Phi-\theta_+)} + C_- e^{ikr\cos(\varphi+\Phi+\theta_-)} +$$

$$+\sum_n C_n \exp\{-ikr\cos[\varphi-(-1)^n\varphi_0-2n\Phi]\} \quad \text{for } kr \gg 1 \tag{35}$$

where

$$f(\varphi,\varphi_0) = \frac{\pi}{2\Phi}\cos\frac{\pi\varphi_0}{2\Phi}\,[F(\varphi_0)]^{-1}\times$$

$$\times\left[\frac{F(\varphi-\pi)}{\sin\frac{\pi(\varphi-\pi)}{2\Phi}-\sin\frac{\pi\varphi_0}{2\Phi}} - \frac{F(\varphi+\pi)}{\sin\frac{\pi(\varphi+\pi)}{2\Phi}-\sin\frac{\pi\varphi_0}{2\Phi}}\right] \tag{36}$$

$f(\varphi, \varphi_0)$ is the amplitude of the diffraction term,

$$C_\pm = 2\sin\frac{\pi^2}{4\Phi}\cos\frac{\pi\varphi_0}{2\Phi}\,M_\Phi\left(2\Phi-\frac{\pi}{2}\right)M_\Phi\left(\frac{\pi}{2}+\theta_+-\theta_-\right)\times$$

$$\times M_\Phi\left(2\Phi+\frac{\pi}{2}+2\theta_\pm\right)M_\Phi\left(\pm\frac{3\pi}{2}+\theta_+-\theta_-\right)[F(\varphi_0)]^{-1}\times$$

$$\times\left[\cos\frac{\pi(\pi+\theta_\pm)}{2}\mp\sin\frac{\pi\varphi_0}{2\Phi}\right]^{-1}$$

$$\text{for: } 0 \leqslant \Phi \mp \varphi < -\arccos\left(\frac{1}{\cos\theta_\pm}\right)^*$$

$$C_\pm = 0$$

$$\text{for: } \Phi \mp \varphi > -\arccos\left(\frac{1}{\cos\theta_\pm}\right) \tag{37}$$

$C_\pm$ are the amplitudes of the surface waves leaving the centre along the boundaries $\pm\Phi$,

$$C_n = (-1)^n F[(-1)^n \varphi_0 - 2n\Phi]\,[F(\varphi_0)]^{-1}$$

$$\text{for } \left|\varphi - 2n\Phi - (-1)^n \left[\arccos\frac{1}{\operatorname{ch}\operatorname{Im}\varphi_0} + \operatorname{Re}\varphi_0\right]\right| < \pi$$

$$C_n = 0$$

$$\text{for } \left|\varphi - 2n\Phi - (-1)^n \left[\arccos\frac{1}{\operatorname{ch}\operatorname{Im}\varphi_0} + \operatorname{Re}\varphi_0\right]\right| > \pi \tag{38}$$

C_n are the amplitudes of the incident wave and of the waves reflected from the boundaries, being of the same nature as the incident wave.

As we remember, φ_0 may be real which corresponds to the scattering of three free particles, or it may be equal to $-\Phi+\theta_-$ or $\Phi-\theta_+$ which corresponds to the scattering on the bound state.

Making use of expression (A.5) we can modify the expressions (35)–(38) to obtain the solution for the case when particles 2, 3 interact *via* the hard rod potential. The component in (35) corresponding to the surface wave along the boundary $\varphi = \Phi$ disappears and for the amplitudes $f(\varphi, \varphi_0)$ and C_- we obtain

$$C_- = 2\sin\frac{\pi^2}{4\Phi}\cos\frac{\pi\varphi_0}{2\Phi}\, M_\Phi\left(2\Phi - \frac{\pi}{2}\right) M_\Phi\left(2\Phi + \frac{\pi}{2} + 2\theta_-\right) \times$$

$$\times \left[M_\Phi\left(\varphi_0 - \Phi - \frac{\pi}{2} + \theta_-\right)\right]^{-1} \left[M_\Phi\left(\varphi_0 - \Phi + \frac{\pi}{2} - \theta_-\right)\right]^{-1} \times$$

$$\times \left[\cos\frac{\pi(\pi+\theta_-)}{2\Phi} + \sin\frac{\pi\varphi_0}{2\Phi}\right]^{-1} \tag{36a}$$

---

* The lines: $\Phi \pm \varphi = -\arccos\left(\frac{1}{\cos\theta\pm}\right)$ can probably be interpreted as the boundaries of the geometric shadow of the surface waves leaving the reaction centre. For the case $\Phi < \frac{4}{\pi}$ which is under consideration only the first inequality in (37) is fulfilled.

$$f(\varphi, \varphi_0) = \frac{\pi}{2\Phi} \cos \frac{\pi\varphi_0}{2\Phi} \left[M_\Phi \left(\varphi_0 - \Phi + \frac{\pi}{2} - \theta_- \right)^{-1} \right] \times$$

$$\times M_\Phi \left(\varphi_0 - \Phi - \frac{\pi}{2} + \theta_- \right)^{-1} \times \left\{ M_\Phi \left(\varphi - \frac{3\pi}{2} - \Phi + \theta_- \right) \times \right.$$

$$\times M_\Phi \left(\varphi - \frac{\pi}{2} - \Phi - \theta_- \right) \left(\sin \frac{\pi(\varphi - \pi)}{2\Phi} - \sin \frac{\pi\varphi_0}{2\Phi} \right)^{-1} -$$

$$- M_\Phi \left(\varphi + \frac{\pi}{2} - \Phi + \theta_- \right) M_\Phi \left(\varphi + \frac{3\pi}{2} - \Phi - \theta_- \right) \times$$

$$\left. \times \left[\sin \frac{\pi(\varphi + \pi)}{2\Phi} - \sin \frac{\pi\varphi_0}{2\Phi} \right]^{-1} \right\}. \tag{37a}$$

In order to translate the solution (35) into the "particlel anguage" we study its behaviour in different directions φ in the (r, φ) plane. At large distances from the origin along the direction $\varphi = -\Phi$, *i. e.* along the line $s_3 = 0$, $t_3 \to -\infty$, the second term in (35) tends to zero and may be neglected as well as the terms expressing reflected waves which are of higher order in r^{-1} than the third term (provided no reflected wave propagates exactly along the boundary). Thus, C_- is the amplitude of probability of finding the scattered wave in the channel 3. In the same way we can show that C_- represents the amplitude of scattering into channel 1.

The asymptotic behaviour of the plane wave in polar coordinates may be described as follows [14]

$$e^{ikr\cos(\varphi-\varphi')} \xrightarrow[r\to\infty]{} (2\pi ikr)^{-\frac{1}{2}} \times$$

$$\times \{ e^{ikr} 2\pi\delta(\varphi - \varphi') + ie^{ikr} 2\pi\delta(\varphi - \varphi' - \pi) \}. \tag{39}$$

Comparing (39) with (35) we see that $f(\varphi, \varphi_0)$ should represent the amplitude of the plane wave travelling in the direction, this plane wave representing the wave function of three free particles (see (29)).

These results could be derived more strictly by using Gerjuoy's approach [15]. According to this theory of many particle scattering, the probability of a particular collision process is given by a current of the scattered wave through a particular surface element of the infinite sphere in the multi-dimensional space. In our case there are three types of elements $rd\varphi$ of a large circle in the (r, φ) plane related to channels 1, 3 and to the dissociation channel, the directions of the normal vectors of these elements being $\varphi = \pm\Phi$ and $(\varphi) < \Phi$ respectively.

The simplest case of our model is the system of three particles with equal masses and with equal potentials of the pairs 1, 2 and 2, 3. The solution is similar to that of an analogous problem with one-dimensional "delta function" potentials [4], namely it consists of the finite number of terms of the same nature as the incident wave and, of course, it could be found without using Maluzhinetz method. If we assume that in the initial state particles 1 and 2 are bound we obtain the solution of the scattering problem in the following form (in the particle coordinates)

$$\Psi(s, t) = \exp\{-\alpha s_3 + iq_3 t_3\} - \frac{3\alpha + i\sqrt{3}\, q_3}{\alpha - i\sqrt{3}\, q_3} \times$$

$$\times \exp\{1/2 s_3(-\alpha + i\sqrt{3}\, q_3) - 1/2 t_3(\sqrt{3}\,\alpha + iq_3)\} +$$

$$+ \left(\frac{3\alpha + i\sqrt{3}\, q_3}{\alpha - i\sqrt{3}\, q_3}\right)\left(\frac{\alpha + i\sqrt{3}\, q_3}{3\alpha - i\sqrt{3}\, q_3}\right) \times$$

$$\times \exp\{-\alpha s_1 + it_1 q_3\} \tag{40}$$

where: $\alpha_3 = \alpha_1 = \alpha$ and $q_3 = \sqrt{k^2 + \alpha^2}$.

The last term was expressed in terms of coordinates s_1 and t_1 in order to show that it represents a state function in channel 1, *i. e.* the reflected wave is at the same time the surface wave leaving the centre along the direction of the boundary $s_1 = 0$. In that case the diffraction term is zero, which is the consequence of the equality of both masses and of the potentials.

In order to obtain (40) from the Maluzhinetz' expression (35) we put

$$\theta_+ = \theta_- = \theta; \ \Phi = \frac{\pi}{6} \text{ and } \varphi_0 = \theta - \frac{\pi}{6}. \tag{41}$$

Using the expression (A.7) we can write the solution in the polar coordinates in the following form

$$\Psi(r\varphi) \exp\left\{-ikr\cos\left(\varphi - \theta + \frac{\pi}{6}\right)\right\} - \sqrt{3}\,\mathrm{tg}\left(\theta - \frac{\pi}{6}\right) \times$$

$$\times \exp\{-ikr\sin(\varphi + \theta)\} + \mathrm{tg}\left(\theta - \frac{\pi}{6}\right)\mathrm{tg}\left(\theta - \frac{\pi}{3}\right) \times$$

$$\times \exp\left\{ikr\cos\left(\varphi - \theta - \frac{\pi}{6}\right)\right\} \tag{42}$$

with reduces to (40) after using the relations

$$k\cos\theta = q_3$$

$$ik\sin\theta = -\alpha. \tag{43}$$

The Maluzhinetz method can be applied to find the solution of equations (21) and (22) for the negative values of $E = k^2$, *i. e.* for the values of three-body energy below the dissociation threshold.

We put

$$k^2 = -m^2 \quad \text{where } m > 0; k = im. \tag{44}$$

We look for the solution of Eq. (21) in the form

$$\Psi(r, \varphi) = \frac{1}{2\pi i} \int_{\gamma'} e^{mr \cos z} s(z+\varphi) dz \tag{45}$$

where the contour γ' is the contour symmetric with respect to the axis Im z and can be obtained from the contour shown in Fig. 3 by shifting its upper branch to the right by $\frac{\pi}{2}$, and its lower branch to the left by $\frac{\pi}{2}$. Further on we shall assume that $k^2 = -m^2 > \max(-\alpha_1^2, -\alpha_3^2)$, *i. e.* that the channels for both the elastic and rearrangement processes are open. We leave the boundary conditions in the same form as given in (23). The relation between the "angles" θ_+ and θ_- and the two-particle constants α_1 and α_3 takes now the form

$$\theta_+ = \operatorname{arc\,tg} \frac{i\alpha_1}{\sqrt{\alpha_1^2-m^2}} = \frac{1}{2i} \ln \frac{\sqrt{\alpha_1^2-m}-\alpha_1}{\sqrt{\alpha_1^2-m^2}+\alpha_1} =$$

$$= \frac{1}{2i} \ln \frac{\sqrt{\alpha_1^2-m^2}-\alpha_1}{-\sqrt{\alpha_1-m^2}-\alpha_1} - \frac{\pi}{2}$$

$$\theta_- = \operatorname{arc\,tg} \frac{i\alpha_3}{\sqrt{\alpha_3^2-m^2}} = \frac{1}{2i} \ln \frac{\sqrt{\alpha_3^2-m^2}-\alpha_3}{-\sqrt{\alpha_3^2-m^2}-\alpha_3} - \frac{\pi}{2} \tag{46}$$

where the quantities under the ln sign on the right-hand side of equalities are positive and the logarithm is understood in the sense of its principal branch. We see that θ_+ and θ_- are now complex numbers with

$$\operatorname{Re} \theta_+, \operatorname{Re} \theta_- = -\frac{\pi}{2}$$

and

$$\operatorname{Im} \theta_+, \operatorname{Im} \theta_- < 0. \tag{47}$$

The functions $\sin\theta = \frac{\alpha}{m}$ are now real and negative, while the functions $\cos\theta = -i\frac{\sqrt{\alpha^2-m^2}}{m}$ are now imaginary with Im $\cos\theta < 0$.

The closed form of the solution $\Psi(r, \varphi)$ is given by the expression (45) with $s(z)$ given by (32). In order to obtain the asymptotic expression of the wave function at large distances from the origin we deform the contour into two lines Re $z = \pm\pi$. Following the procedure applied in Ref. [11c] we can extract the terms corresponding to elastic and rearrangement scatterings. The remaining integral can be treated with the saddle point method and asymptotically appears to be exponentially decreasing in all directions $(\varphi) < \Phi$.

The author is indebted to Professor J. Stecki for his encouragement and interest in this work.

Note added in manuscript: After this manuscript was submitted for publication it came to the author's knowledge that a somewhat similar problem had been studied by H. M. Nussenzweig, *Proc. Roy. Soc.*, **264A**, 408 (1961). The model discussed by Nussenzweig can be regarded as a special case of the model considered in this paper when taking: $m_1 = \infty$, $m_2 = m_3$, $V_1 = V_{\text{hard core}}$ and assuming that the motion is confined to the region $s_1 < 0$, $s_3 > 0 \left(2\Phi = \frac{3\pi}{4}\right)$. In this case the expressions for the scattering amplitudes can be written in terms of elementary functions (see Equations (A.6) of this paper).

APPENDIX A

Here we collect the properties and the integral representations of the special function $M(z)$ introduced by Maluzhinetz in his studies on the diffraction problems [11c], [11d], [12]. In the cited literature the special function called by us $M_\Phi(z)$ is notated $\Psi_\Phi(z)$.

The definition and integral representations

$$M_\Phi(z) = \prod_{n=1}^{\infty} \prod_{m=1}^{\infty} \left[1 - \left(\frac{z}{2\Phi(2n-1) + \frac{\pi}{2}(2m-1)}\right)^2\right]^{(-1)^{m+1}} \tag{A.1}$$

$$M_\Phi(z) = \exp\left[\frac{i}{8\Phi} \int_0^z d\mu \int_{-i\infty}^{i\infty} \operatorname{tg} \frac{\pi v}{4\Phi} \frac{dv}{\cos(v-\mu)}\right] \tag{A.2}$$

$$M_\Phi(z) = \exp\left[-\frac{1}{2} \int_0^\infty \frac{(\operatorname{ch} zs - 1)}{s \operatorname{ch}\left(\frac{\pi}{2} s\right) \operatorname{sh} 2\Phi s} ds\right]. \tag{A.3}$$

The integral in (A.3) is convergent for $|\text{Re}\, z| < 2\Phi + \frac{\pi}{2}$.

The zeros and poles nearest to the point $z = 0$ are

$$z = \pm\left(\frac{\pi}{2} + 2\Phi\right)$$

$$z = \pm\left(\frac{3\pi}{2} + 2\Phi\right) \quad \text{respectively} \quad (\Phi > 0). \tag{A.4}$$

For $|\text{Im}\, z| \to \infty$ we have $M_\Phi(z) = 0\left[\exp\left|\frac{\pi \,\text{Im}\, z}{8\Phi}\right|\right]$. (A.5)

For the values of Φ such that $4\frac{\Phi}{\pi}=\frac{n}{m}$, where n and m are integers, we have

$$M_\Phi(z)=\prod_{k=1}^{m}\prod_{l=1}^{n}\left(\frac{\cos 1/2a(k,l)}{\cos 1/2z/n-a(k,l)}\right)^{(-l)} \quad \text{for } n \text{ odd}.$$

$$M_\Phi(z)=\prod_{k=1}^{m}\prod_{l=1}^{n}\exp\left\{\frac{(-1)^l}{\pi}\int_{a(k,l)}^{a(k,l)+\frac{z}{n}} u\operatorname{ctg} u\,du\right\} \quad \text{for } n \text{ even} \tag{A.6}$$

where $a(k,l)=\frac{\pi}{2}\left(\frac{2l-1}{n}-\frac{2k-1}{m}\right)$, and the fraction $\frac{n}{m}$ is irreducible.

Other properties of the functions $M_\Phi(z)$ are

$$\frac{M_\Phi(z+2\Phi)}{M_\Phi(z-2\Phi)}=\operatorname{ctg} 1/2\left(z+\frac{\pi}{2}\right) \tag{A.7}$$

$$M_\Phi\left(z+\frac{\pi}{2}\right)M_\Phi\left(z-\frac{\pi}{2}\right)=M_\Phi^2\left(\frac{\pi}{2}\right)\cos\frac{\pi z}{4\Phi} \tag{A.8}$$

$$M_\Phi(z+\Phi)M_\Phi(z-\Phi)=[M_\Phi(\Phi)]^2M_{\Phi/2}(z). \tag{A.9}$$

The logarithmic derivative of the function $M_\Phi(z)$ is described in detail in Ref. [11a]. The tables of the function $M_\Phi(z)$ were constructed by Zavadskij and Sakharova [12].

REFERENCES

[1] J. B. McGuire, *J. Math. Phys.*, **5**, 622 (1964).
[2] E. Lieb, H. Koppe, *Phys. Rev.*, **116**, 367 (1959).
[3] W. Bierter, K. Dietrich, *Z. Phys.*, **202**, 74 (1967); W. Bierter, K. Dietrich, *Nuovo Cimento*, **52A**, 1208 (1967).
[4] R. Dodd, *J. Math. Phys.*, **11**, 207 (1970).
[5] H. Eyring, H. Gershinovitz, C. E. Sun, *J. Chem. Phys.*, **3**, 786 (1935).
[6] D. W Jepsen, J. O. Hirschfelder, *J. Chem. Phys.*, **30**, 1032 (1959).
[7] K. Lipszyc, J. Stecki, *Bull. Acad. Polon. Sci., Ser. Sci. Chim.*, **18**, 541 (1970).
[8] K. Lipszyc, *J. Math. Phys.*, (in press).
[9] G. V. Skorniakov, K. A. Ter-Martirosian, *Zh. Eksper. Teor. Fiz.*, **131**, 775 (1956).
[10] G. Flamand, *Mathematical Theory of Non-Relativistic Two- and Three-Particle Systems with Point Interactions in Applications of Mathematics to Problems in Theoretical Physics*, ed F. Lurcat, Gordon and Breach Sci. Publ. Inc., New York 1967.
[11a] G. D. Maluzhinetz, *Akust. Zh.*, **1**, 144 (1955).
[11b] G. D. Maluzhinetz, *Dokl. Akad. Nauk SSSR*, **118**, 1099 (1958).
[11c] G. D. Maluzhinetz, *Dokl. Akad. Nauk SSSR*, **121**, 436 (1958).
[11d] G. D. Maluzhinetz, *Ann. Phys.*, (Germany) **6**, 107 (1960).
[12] V. J. Zavadskij, M. P. Sacharova, *Akust. Zh.*, **13**, 60 (1967).
[13] M. S. Bobrovnikov, R. P. Starovoitova, *Izv. VUZ Fiz.*, **6**, 168 (1963).
[14] P. M. Morse, H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill Co., New York 1953, p. 1379.
[15a] E. Gerjuoy, *Ann. Phys.*, **5**, 58 (1958).
[15b] E. Gerjuoy, *Phys. Rev.*, **109**, 1806 (1958).