SOME SIMPLE MODELS OF ANISOTROPIC FERMI LIQUIDS. I

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The anisotropic Fermi liquid with a simple ellipsoidal band is considered by means of the microscopic approach. It is shown that the linear electrodynamics of the system can be obtained from that for the isotropic case by means of a suitable transformation of the wave vector provided that we restrict ourselves to the consideration of some class of effective interactions. It is also shown that the reaction of the system can be split into the longitudinal and transversal reactions. We study both collisionless reactions in the quasistatic and quasi-homogeneous limit in the radio-frequency as well as the infrared region. Some of our results are more general, for example we prove that the electron-phonon interaction does not change the static properties of the system for any band structure.

1. Introduction. Specification of the model

The purpose of the series of papers beginning with this article is to consider solvable models of charged anisotropic Fermi liquids from the microscopic point of view. The word "solvable" means that the results will not be restricted to pretty general formulae or even integral equations with kernels unknown except for their symmetry properties. Our main interests will be confined to the consideration of the electrodynamics of systems, such as the problem of the phonon renormalization of the electron properties when the frequency of an external field is much smaller than the maximal phonon frequency. On the other hand, this "radio-frequency electrodynamics" will be confronted with that for external frequency much greater than the maximal phonon frequency, *i. e.* with the infrared electrodynamics.

Our methods will be analogous to the methods developed by Luttinger and Nozières [1, 2] and Legget [3] whereas our notation will be almost identical with that used in the above papers. These methods were used by us to obtain radio-frequency (rf) electrodynamics for the isotropic model of metals (with the application to policrystalline samples of alkali metals), [4]. It should be noted that some important aspects of the theory of anisotropic

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Fermi liquids were discussed by Jones and McClure [5]. On the other hand, such problems as the microscopic approach to screening or phonon renormalization (important in the rf limit) remain undiscussed for the anisotropic systems up to the present time. This statement is less precise as concerns the second problem since its algebraic scheme was constructed recently [6, 7]. It was performed for the simple model of ferromagnetic Fermi liquids where the spin-dependent level density (treated as a diagonal spin matrix) does not commute with the spin-symmetric matrices of the effective interactions of quasiparticles. Moreover, the effective quasiparticle interaction with the contribution of the electron-phonon interaction (EPI) does not commute with the analogous quantity of purely Coulomb character. Let us mention that the generalization to level densities dependent on the band index as well as the replacement of effective interactions used in [6] by matrix with respect to band indices with a more general two-quasimomenta dependence does not change any general algebraic relation of the paper [6]. Taking into account this fact we find that our general formulae remain valid also in the case generalized in the above manner. Hence we obtain that EPI does not change the static properties of the system even under assumptions generalized as above, because this fact is of a purely algebraic character (*cf.* [6]).

Let us formulate the model which will be considered in the present paper. This model is identical with that considered by Alodjianz [8] in a phenomenological way. We have here a single ellipsoidal band of a simple form and the electrons interact by momentum conserving interaction. The Fermi surface has also the ellipsoidal form. The symmetry group of this surface is D_{2h}. The most general quasiparticle interaction compatible with this group in the spin-direct as well as the spin-exchange channel (we assume the spin-orbit coupling to be unimportant) has the form: $f(\tilde{\boldsymbol{p}};\ \tilde{\boldsymbol{p}}';\ p_x p'_x,\ p_y p'_y,\ p_z p'_z)$ in the reference frame of the symmetry axes. Here $\boldsymbol{p} \equiv [p_x, p_y, p_z]$ and $\boldsymbol{p}'$ denote momenta of quasiparticles, $\tilde{\boldsymbol{p}}$ is defined as $[|p_x|, |p_y|, |p_z|]$ *etc.*, and the function f is invariant with respect to the mutual substitution $\tilde{\boldsymbol{p}} \rightleftarrows \tilde{\boldsymbol{p}}'$. If we consider the above effective interactions then even the simpler problems such as the relations between these quantities and the scattering amplitudes, the expressions for basic dressed vertices in the "ω" or "k" limits, *etc.*, can be formulated only in th·form of integral equations with unknown kernels. To avoid this situation we assume, according to [8], that the effective interactions depend only on the "angle" between two "vectors" $\boldsymbol{u}$, $\boldsymbol{u}'$ the components of which are determined by the momenta $\boldsymbol{p}$, $\boldsymbol{p}'$ in the reference frame of symmetry axes in the following way $u_i = p_i(M/m_i)^{1/2}$, $i = (x, y, z)$. Here m_i denotes the eigenvalue of the lattice mass tensor in the i-th direction whereas M is the cubic root of the determinant if this mass tensor. The choice M causes that the Jacobian of the linear transformation $\boldsymbol{p} \to \boldsymbol{u}$ is equal to unity. It is clear that $\boldsymbol{u}$ and $\boldsymbol{u}'$ do not transform as vectors and their mutual "angle" is not an invariant of the orthogonal group. On the other hand, the effective interactions depend only on the cosine of the above angle as a result of the symmetry with respect to $\boldsymbol{p} \rightleftarrows \boldsymbol{p}'$.

It is well-known that a Fermi surface in the form of a single ellipsoid does not occur in practice. Moreover, the majority of simple metals have cubic symmetry. In this case we have three equivalent perpendicular axes, the ellipsoid degenerates into the sphere and we come back to the theory of isotropic systems developed in [1–4, 6, 7]. On the other

hand, for the hexagonal metals the corresponding ellipsoid is two-axial and its circular sections are perpendicular to the hexagonal (or trigonal) axis. It should be emphasized that in some semimetals the Fermi surface consists of a few ellipsoids. As was shown recently the zero sound can appear in these semimetals [9]. From this point of view, our paper can be treated as a necessary step toward considering the more physical cases when a few ellipsoids occur.

2. *The Ward identities and the basic formulae*

Let us consider a system with Hamiltonian such that *i*) the Hamiltonian of noninteracting particles is of the type of the kinetic energy, *i. e.* $\boldsymbol{p}$-concerving, but the dispersion curve $\varepsilon(\boldsymbol{p})$ is anisotropic and even more general than was postulated in the previous chapter *ii*) the interaction term is also $\boldsymbol{p}$-conserving. This system is simpler than that considered in [5]; the possible application of the model is restricted to the case of a Fermi surface existing in one band only whereas the contribution of inter-band transitions is negligible. It can be easily seen that three Ward identities can be obtained for our model using the methods developed in [1]. In the notation of [1] one can write

$$1-\partial G_p/\partial\mu = \tilde{T}_p^{ok}, \tag{1}$$

$$\tilde{T}_p^{ak} = v_p^a+(\partial G_p/\partial p_a),\ a = x, y, z; \tag{2}$$

$$\tilde{T}_p^{o\omega} = 1-(\partial G_p/\partial\zeta_l). \tag{3}$$

Here $p \equiv (\boldsymbol{p}, \zeta_l)$, $v_p^a \equiv \partial\varepsilon(\boldsymbol{p})/\partial p_a$ and $\tilde{T}^o$, $\tilde{T}^a$ are proper vertex functions induced by the scalar and vectorial (v_p^a) bare vertices respectively and μ is the chemical potential. The superscripts "k" or "ω" denote that the vertices correspond respectively to a weakly inhomogeneous time-independent field or to a homogeneous field slowly varying in time. Note also that G_p denotes the mass operator whereas ζ_l is the imaginary frequency of Green's functions of the Matsubara type. All quantities mentioned above are defined in detail in [1] and we do not repeat these definitions here. The same situation will occur with the remaining basic quantities of the theory of Fermi liquids. Using momentum conservation, analogously as in [1], it can be shown that

$$p_a\left(1-\frac{\partial G_p}{\partial\zeta_l}\right) = p_a+\sum_{p'\sigma'}\tilde{\Gamma}_{pp'}^{c\omega}R_{p'}^{\omega}p'_a \tag{4}$$

($\tilde{\Gamma}_{pp'}^{c\omega}$ denotes the proper part of the four-point function in the ω-limit whereas $R_{p'}^{\omega}$ — the regular part of squared Green's function; $\tilde{\Gamma}^{c\omega}$ does not contain the contribution of EPI). The formula (4) can be rewritten in a form similar to (1–3) but only provided the square dispersion law of $\varepsilon(\boldsymbol{p})$ is fulfilled. Then

$$V_p^a\left(1-\frac{\partial G_p}{\partial\zeta_l}\right) = \tilde{T}_p^{a\omega}. \tag{5}$$

Introducing the renormalized vertices τ [1] we have on the Fermi surface

$$\tau_p^{0k} = 1-\partial E_p/\partial\mu, \quad \tau_p^{ak} = \partial E_p/\partial p_a \equiv V_p^{ca},$$
$$\tau_p^{0\omega} = 1, \quad \tau_p^{a\omega} = v_p^a. \tag{6}$$

Here E_p denotes the excitation energy with the contribution of the Coulomb interaction and the last relation is fulfilled only for the square dispersion law $\varepsilon(\boldsymbol{p})$. It should be emphasized that the renormalization factor Z_p^c, denoting the discontinuity of the particle density on the Fermi surface can be $\boldsymbol{p}$-dependent. The vertex V_p^{ca} near the Fermi surface is equal to p_a/m_{ca}, where m_{ca} is the eigenvalue of the mass tensor with the contribution of the Coulomb interelectron interaction; this term, as it is isotropic, cannot change the direction of the symmetry axes. The relation between the four-point functions $\tilde{\Gamma}^{c\omega}$ and $\tilde{\Gamma}^{ck}$ has the same form as in [1], *i. e.*

$$Z_p^c\tilde{\Gamma}^{c\omega}_{p,p'}Z_{p'}^c - Z_p^c\tilde{\Gamma}^{ck}_{p,p'}Z_{p'}^c =$$
$$\sum_{p''\sigma} Z_p^c\tilde{\Gamma}^{c\omega}_{p,p''}Z_{p''}^c\delta(E_{p''}-\mu)Z_{p''}^c\tilde{\Gamma}^{ck}_{p'',p'}Z_{p'}^c. \tag{7}$$

Passing from summation to integration in (7), performing the change of variables $p \to \boldsymbol{u}$ such that $u_a = p_a(M_c/m_{ca})^{1/2}$, $M_c = (m_{cx}m_{cy}m_{cy})^{1/3}$, we find that the relation between dimensionless amplitudes f^ω and f^k remins the same as for the isotropic system provided that $Z_p^c\tilde{\Gamma}^{c\omega}_{p,p'}Z_{p'}^c$ depends only on the angle between $\boldsymbol{u}$ and $\boldsymbol{u}'$. Here the dimensionless amplitudes are defined by

$$\begin{Bmatrix} f^\omega(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}') \\ f^k(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}') \end{Bmatrix} = \frac{u_F M_c}{\pi^2} Z_p^c Z_{p'}^c \begin{Bmatrix} \tilde{\Gamma}^{c\omega}_{p,p'} \\ \tilde{\Gamma}^{ck}_{p,p,} \end{Bmatrix}, \tag{8}$$

with $u_F^2 = u_x^2+u_y^2+u_z^2$, $\hbar \equiv 1$. The Legendre amplitudes of angle-dependent functions will be defined according to [4, 6, 10]. This definition differs by the factor $(2l+1)$ from that usually used *e. g.* in [3]. The Legendre amplitudes of the spin-direct and the spin-exchange part of f^ω will be denoted by A_l and B_l respectively. Note that the relations between the l-th Legendre amplitude of the spin-direct and the spin-exchange part of f^k and A_l, B_l respectively remain the same as for isotropic systems. Let us introduce the proper correlation functions of vertices, $\tilde{S}^{ab}(k, \omega)$, [1]. Here $\boldsymbol{k}$, ω denote the wave-vector and the frequency of the external field and a, b are equal 0, 1, 2 or 3; further 1, 2, 3 will stand for x, y, z. Using the Ward identities we can prove that

$$\tilde{S}_\omega^{ab} = 0; \; \tilde{S}_k^{00} = \partial N/\partial\mu; \; \tilde{S}_k^{ab} = N\delta_{ab}/m_a, \; a, b > 0; \tag{9}$$

where N denotes the total number of particles. It should be emphasized that $S_\omega^{ab} = 0$ for $a, b > 0$ as a result of (5). It can be proved that the spin autocorrelation function determines the Pauli susceptibility in the k-limit whereas this function vanishes in the ω-limit.

Applying the methods developed in [10] one can obtain the basic formulae for this model. The situation can be summarized as follows:

i) the static properties preserve their form in comparison to isotropic systems if we substitute u_F instead of the Fermi momentum and M_c instead of the isotropic effective mass; particularily $N = u_F^3/3\pi^2$, (we assume further that the system has unit volume)

ii) the stability conditions as well as the Leggett inequality [3] preserve their form

iii) we have

$$m_{ca} = m_a(1+A_1). \tag{10}$$

Hence $(M_c/m_{ca}) = (M/m_a)$ and this is why there is no contradiction between the transformations $\boldsymbol{p} \to \boldsymbol{u}$ used in the first chapter and here.

Applying methods analogous to those in papers [3, 11, 12] we find that

$$\frac{V_{np}}{Z_{np}} = \frac{V_{np}^c}{Z_{np}^c}, \tag{11}$$

where V_{np}, Z_{np} denote respectively the velocity and the discontinuity of the particle density on the Fermi surface with the contribution of EPI, n being the band index. For our model, omitting the band index and taking into account that $V_p^{ca} = p_a/m_{ca}$ we find that $V_p^a = p_a/m_a^*$ and that the ratio

$$\frac{Z_p}{Z_p^c} = \frac{m_{ca}}{m_a^*} = \frac{M_c}{M^*}, \qquad M^* = (m_1^* m_2^* m_3^*)^{1/3}, \tag{12}$$

is p-independent. Hence, in the transformation $\boldsymbol{p} \to \boldsymbol{u}$ the ratio $(M^*/m_a^*)^{1/2}$ can also be used. The other identities for the system with important EPI contribution will be given in the next chapter. They will be very similar to those obtained for isotropic systems [3, 11, 12].

3. *The collisionless long-wavelength electrodynamics of the system*

The linear reaction of the system to the external electromagnetic field of wave-vector $\boldsymbol{k}$ and frequency ω is described by the correlation functions $S^{ab}(\boldsymbol{k}, \omega)$ where $a, b = 0, 1, 2, 3$, [1]. Written in terms of the proper correlation functions $S^{ab}(\boldsymbol{k}, \omega)$ they have the same form as for the isotropic systems. We have

$$S^{ab}(\boldsymbol{k}, \omega) = \tilde{S}^{ab}(\boldsymbol{k}, \omega) + \frac{\dfrac{4\pi e^2}{|\boldsymbol{k}|^2}\tilde{S}^{a0}(\boldsymbol{k}, \omega)\tilde{S}^{b0}(\boldsymbol{k}, \omega)}{1 - \dfrac{4\pi e^2}{|\boldsymbol{k}|^2}\tilde{S}^{00}(\boldsymbol{k}, \omega)}. \tag{13}$$

For parabolic $\varepsilon(\boldsymbol{p})$ one can represent $S^{ab}(\boldsymbol{k}, \omega)$ by the expression containing only the integrations near the Fermi surface, for more general $\varepsilon(\boldsymbol{p})$ this is possible only for $S^{00}(\boldsymbol{k}, \omega)$, since (5) is then not fulfilled. Under the above assumptions

$$\tilde{S}^{ab}(\boldsymbol{k}, \omega) = I_{ab}(\boldsymbol{k}, \omega) =$$

$$= \tilde{\mathrm{Tr}}\,\{v^a (Z^c)^{-1} g^n(\boldsymbol{k}, \omega)\,[1 + \tilde{\Gamma}(\boldsymbol{k}, \omega) g^n(\boldsymbol{k}, \omega)]\,(Z^c)^{-1} v^b\}, \tag{14}$$

where $v^0 \equiv 1$, $g^n(\boldsymbol{k}, \omega)$ is the product of two polar parts of Green's functions with the energy-momentum transfer $\boldsymbol{k}, \omega$, divided by $(2\pi)^4$. The multiplication in the curly bracket has an operatorial character with summation over intermediate Galilean four — momentum and spin; it should be taken into account that all operators appearing there are diagonal with the exception of the four-point function $\tilde{\Gamma}(\boldsymbol{k}, \omega)$. The "trace with tilda" operation is defined as a twofold integral over external four-momenta with the summation over two external spin indices (*cf.* [4, 6]).

If $|\omega|$ is much greater than the maximal phonon frequency then $g^n(\boldsymbol{k}, \omega)$ has a simple form. In the opposite limit one has to perform the additional transformation of proper correlation functions [3]. A more general algebraic scheme than that developed in [3] needs to be applied as a result of a possible uncommutativity of diagonal operators Z, Z^c with respect to $\tilde{\Gamma}$. The more general scheme developed in [6] is here quite satisfactory. After rather long but very typical calculations (*cf.* [3] and [4]) we find, using (8) and (12), that $I_{ab}(\boldsymbol{k}, \omega) = \Phi_{ab} + I'_{ab}(\boldsymbol{k}, \omega)$, where

$$\Phi_{ab} = \frac{(M_c/M^*)^2 N^*}{1+A_{r(a)}} \left\{ \frac{1+F_{r(a)}}{1+A_{r(a)}} - \frac{M^*}{M_c} \right\} \langle v^a v^b \rangle, \tag{15}$$

$$I'_{ab}(\boldsymbol{k}, \omega) = N^*(1+F_1)^{r(a)+r(b)} \langle V^a Q(\boldsymbol{k}, \omega) \left[1 - FQ(\boldsymbol{k}, \omega)\right]^{-1} V^b \rangle. \tag{16}$$

Here $V^a = 1$ for $a = 0$ and p_a/m_a for $a < 0$, $r(a) \equiv 1 - \delta_{a0}$, $N^* = M^* u_F/\pi^2$. The multiplication under the symbol has a matrix character with the average spherical angles of $\boldsymbol{u}$-vector connected with the intermediate momentum, the summation over intermediate spins was already performed and disappears in (15) and (16). All quantities appearing there are taken on the Fermi surface, *e.g.* $V_a = \hat{u}_a u_F (M^* m_a^*)^{-1/2}$, $a > 0$, $v^a = \hat{u}_a u_F (M m_a)^{-1/2}$, $a > 0$, with $\hat{u}_a \equiv u_a/u_F$. According to the definition of the matrix multiplication and the definition of $\widetilde{\mathrm{Tr}}$ the symbol $\langle \ldots v \rangle$ denotes the double average over spherical angles of $\boldsymbol{u}$-vectors defined by external momenta. The quantity F denotes the spin-direct part of the dimensionless effective interaction of quasiparticles with the contribution of EPI. In the proof of the formulae (15), (16) it was assumed that this quantity defined as

$$F(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}') = \tfrac{1}{2} N^* \sum_{\beta} \tilde{\Gamma}^{\omega}_{\alpha\beta,\alpha\beta}(00|\boldsymbol{p}, \boldsymbol{p}') Z_p Z_{p'} \tag{17}$$

depends only on $\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}'$, where α, β denote spin indices and the function under summation was defined in [3]. The remaining undefined symbol, $Q(\boldsymbol{k}, \omega)$, is determined as the diagonal operator with the elements $\boldsymbol{kV}/(\omega - \boldsymbol{kV}) = \tilde{\boldsymbol{k}}\tilde{\boldsymbol{V}}/(\omega - \tilde{\boldsymbol{k}}\tilde{\boldsymbol{V}})$, where $\tilde{\boldsymbol{V}} = (u_F/M^*)\hat{\boldsymbol{u}} \equiv \tilde{V}\hat{\boldsymbol{u}}$ and $\tilde{k}_a = k_a (M^*/m_a^*)^{1/2}$ (we gave the definition of $\tilde{k}_a$-components only in the reference frame of symmetry axes of our system, but the generalization to any other reference frame is obvious).

Since electrons are not produced (or annihilated) in acts of EPI, $\Phi_{00} = 0$, [3], and hence, from (15) we get

$$(M^*/M_c) = (1+F_0)/(1+A_0). \tag{18}$$

Analogously, taking into account that EPI is spin-conserving one can obtain

$$(M^*/M_c) = (1+G_0)/(1+B_0), \tag{19}$$

where B_l, $l = 0, 1, \ldots$ is the l-th Legendre amplitude of the exchange part of $f^{c\omega}$, whereas G_l is determined in the same manner by the dimensionless effective interaction with the contribution of EPI. It can be proved in the same way that substituting into (18), (19) the suitable ratios of amplitudes with higher l we obtain suitable Leggett's inequalities. Taking into account (10) and the relation between N and u_F we obtain from (15) for, a $a, b > 0$

$$\Phi_{ab} = N\left[\frac{1+F_1}{m_a^*} - \frac{1}{m_a}\right]\delta_{ab} \leqslant 0. \tag{20}$$

It can be verified that Φ_{a0} for $a > 0$ vanishes as a result of time-reversal invariance even in cases more general than those considered here.

Let us compare the obtained results for $I'_{ab}(\boldsymbol{k}, \omega)$ with the analogous results for the isotropic systems (*cf.* [4]). Taking into account he form of the $Q(\boldsymbol{k}, \omega)$-operator and V^a vertices we find that our $I'_{ab}(\boldsymbol{k}, \omega)$ is equal to the same quantity for isotropic systems provided that we substitute here instead of $\boldsymbol{V}$ and $\boldsymbol{k}$ the "vectors" $\tilde{\boldsymbol{V}}$ and $\tilde{\boldsymbol{k}}$. This result can be written in the form

$$I'_{ab}(\boldsymbol{k}, \omega) = \left(\frac{M^*}{m_a^*}\right)^{1/2}\left(\frac{M^*}{m_b^*}\right)^{1/2}\tilde{I}'_{ab}(\tilde{\boldsymbol{k}}, \omega), \qquad a, b \geqslant 0, \tag{21}$$

where $\tilde{I}'_{ab}(\boldsymbol{k}, \omega)$ denotes the function for isotropic systems with $\tilde{V}$ substituted instead of $V = p_F/m$ and m_0^* is defined formally as M^*. Using the identities proved in the Appendix of our paper [4] and the formula (21) we can write

$$\sum_{a=1}^{3} k_a I'_{ab}(\boldsymbol{k}, \omega) = \omega I'_{0b}(\boldsymbol{k}, \omega) - \frac{N(1+F_1)k_b r(b)}{m_b^*}. \tag{22}$$

Let us now obtain the relations resulting from the symmetry properties of the functions $\tilde{I}'_{ab}(\boldsymbol{k}, \omega)$. For isotropic systems $\tilde{I}'_{ab}(\boldsymbol{k}, \omega) = k_a k_b U(|\boldsymbol{k}|, \omega) + \delta_{ab} R(|\boldsymbol{k}|, \omega)$, $\tilde{I}'_{a0}(\boldsymbol{k}, \omega) = k_a V(|\boldsymbol{k}|, \omega)$ whereas $\tilde{I}'_{00}(\boldsymbol{k}, \omega)$ depends only on $|\boldsymbol{k}|, \omega$. Using (21) and (22) we can obtain the relations between R and U, $\tilde{I}'_{00}$ and between V and $\tilde{I}'_{00}$. We have

$$R(\tilde{k}, \omega) + \tilde{k}^2 U(\tilde{k}, \omega) = \frac{\omega^2}{k^2}\tilde{I}'_{00}(\tilde{k}, \omega) - \frac{N(1+F_1)}{M^*},$$

$$V(\tilde{k}, \omega) = \frac{\omega}{\tilde{k}^2}\tilde{I}'_{00}(k, \omega). \tag{23}$$

Performing considerations analogous to [1] we can obtain that the relation between the induced current and the external electric field $\boldsymbol{E}'$ with the wave vector $\boldsymbol{k}$ and frequency ω can be written as follows

$$J_a = \frac{ie^2}{\omega}\sum_{b=1}^{3}\left[S^{ab}(\boldsymbol{k}, \omega) + \frac{N}{m_a}\delta_{ab}\right]E'_b, \qquad a > 0. \tag{24}$$

On the other hand, the induced charge is given by

$$\varrho = \frac{ie^2}{\omega} \sum_{b=1}^{3} S^{0b}(\boldsymbol{k}, \omega) E'_b. \tag{25}$$

The total electric field $\boldsymbol{E} = \boldsymbol{E}' - 4\pi i \boldsymbol{k}\varrho/|\boldsymbol{k}|^2$ and the relations (24) and (25) can be rewritten in terms of $\boldsymbol{E}$ as follows (see Appendix A)

$$J_a = \frac{ie^2}{\omega} \sum_{b=1}^{3} \left[\tilde{S}^{ab}(\boldsymbol{k}, \omega) + \frac{N}{m_a} \delta_{ab} \right] E_b, \tag{26}$$

$$\varrho = \frac{ie^2}{\omega} \sum_{b=1}^{3} \tilde{S}^{0b}(\boldsymbol{k}, \omega) E_b. \tag{27}$$

The proof of the formulae (26), (27) from (24), (25) is the result of the identity (22). It can be easily seen that owing to this identity the continuity equation and the gauge invariance are preserved. The relation (26) defines the conductivity tensor $\sigma_{ab}(\boldsymbol{k}, \omega)$. Applying (20–23) and (26) we can write

$$\sigma_{ab}(k, \omega) = \frac{ie^2}{\omega} \left\{ \left[\frac{M^*}{m_a^*} \delta_{ab} - \frac{(M^*)^2 k_a k_b}{m_a^* m_b^* \tilde{k}^2} \right] \left[\frac{N(1+F_1)}{M^*} + R(\tilde{k}, \omega) \right] + \right.$$

$$\left. + \frac{\omega^2 (M^*)^2 k_a k_b}{\tilde{k}^4 m_a^* m_b^*} \tilde{I}'_{00}(\tilde{k}, \omega) \right\}. \tag{28}$$

Analogously, (27) can be rewritten as follows

$$\varrho = ie^2 \tilde{I}'_{00}(\tilde{k}, \omega) (\tilde{E}\tilde{k})/\tilde{k}^2, \tag{29}$$

where the components of $\tilde{\boldsymbol{E}}$ are defined by $\boldsymbol{E}$ in the same manner as $\tilde{\boldsymbol{k}}$ by $\boldsymbol{k}$. The scalar product $(\tilde{E}\tilde{k})$ in an arbitrary reference frame has the form

$$M^* \sum_{a=1}^{3} \sum_{b=1}^{3} \left(\frac{1}{m^*} \right)_{ab} E_a k_b, \tag{30}$$

where $(1/m^*)$ denotes the tensor of reciprocal effective mass and $\tilde{k}^2 \equiv (\tilde{k}\tilde{k})$.

For the longitudinal electric field the first term in the curly bracket in (28) is uneffective. On the other hand, the action of the second term is such that the diagonal tensor

$$\sigma_{ab}^{||}(\boldsymbol{k}, \omega) = \frac{ie^2\omega}{k^2} I'_{00}(\tilde{k}, \omega) \left(\frac{M^*}{m_a^*} \right) \delta_{ab} \tag{31}$$

can be defined as the longitudinal tensor of conductivity. It is clear that for longitudinal $\boldsymbol{E}$ the external electric field is also longitudinal and the relation between both fields obtained using (29) has the form

$$\boldsymbol{E}\left[1-\frac{4\pi e^2}{k^2}\tilde{I}'_{00}(\tilde{k},\omega)\right]\equiv \boldsymbol{E}\varepsilon(\boldsymbol{k},\omega)=\boldsymbol{E}'. \tag{32}$$

Hence we find that the quantity in the square bracket above is to be treated as the dielectric function for longitudinal fields. This quantity does not have a tensorial character even though it is anisotropic. The comparison of (31) and (32) shows that the relation $\varepsilon''=1+4\pi i\sigma''/\omega$ is not fulfilled for anisotropic systems (*cf.* [13]).

Let us consider the reaction to fields which are analogous to transversal ones in the isotropic case. They are characterized by the condition that the induced charge vanishes. This last can be written from (29) as $\tilde{\boldsymbol{E}}\perp\tilde{\boldsymbol{k}}$, which is the condition unequivalent to $\boldsymbol{E}\perp\boldsymbol{k}$ even though $\tilde{\boldsymbol{E}}\parallel\tilde{\boldsymbol{k}}$ is equivalent to $\boldsymbol{E}\parallel\boldsymbol{k}$. Such fields will be called simply transversal in the following. It can be easily seen that an arbitrary vector $\boldsymbol{E}$ can be uniquely decomposed into a longitudinal and a transversal part in the above meaning. The transversal part of the vector $\boldsymbol{E}$ is given by $\boldsymbol{E}-\boldsymbol{k}(\tilde{\boldsymbol{E}}\tilde{\boldsymbol{k}})/\tilde{\boldsymbol{k}}^2$ whereas the longitudinal part by $\boldsymbol{k}(\tilde{\boldsymbol{E}}\tilde{\boldsymbol{k}})/\tilde{\boldsymbol{k}}^2$. For the transversal fields the second term in the curly bracket in (28) is uneffective whereas the first term acts as

$$\sigma^{\perp}_{ab}(\boldsymbol{k},\omega)=\frac{ie^2M^*}{\omega m_a^*}\left[\frac{N(1+F_1)}{M^*}+R(\tilde{k},\omega)\right]\delta_{ab}. \tag{33}$$

This quantity can be determined as the transversal tensor of conductivity.

As we have shown the tensor $\sigma_{ab}(\boldsymbol{k},\omega)$ is the sum of two terms such that first of them is uneffective for longitudinal fields whereas the second one is uneffective for transversal fields. This shows that the reaction of the system on the electromagnetic field can be split into a longitudinal and transversal reactions. This is rather a peculiar property of the systems considered. It should be emphasized that the reaction of the system in the infrared region (*i.e.* for frequencies much greater than the maximal phonon frequency) can be obtained from our results by simple specification. If we substitute $\tilde{\Gamma}^{\omega}(00)\to\tilde{\Gamma}^{c\omega}$ and $m_a^*\to m_{ca}$ then we obtain the results for the infrared region.

4. Quasistatic and quasihomogeneous limit. Concluding remarks

Let us apply our results for a spherical Fermi surface in order to obtain the reaction of our system using (21) and (28). Let us discuss the reaction to the longitudinal field. Then the reaction is described completely by the function $\tilde{I}'_{00}(\tilde{k},\omega)$, which is simply related to $\varepsilon(\boldsymbol{k},\omega)$. In the quasistatic limit (*i. e.* for $\omega\ll k\tilde{V}$) we have

$$\varepsilon(k,\omega)\cong 1+\frac{4\pi e^2}{k^2}\left(\frac{\partial N}{\partial\mu}\right)\left[1+\frac{\pi i}{2}\left(\frac{\omega}{\tilde{k}\tilde{V}}\right)\frac{1}{1+F_0}\right], \tag{34}$$

where $(\partial N/\partial\mu)=u_FM^*/\pi^2(1+F_0)$. Using (18) we find that the results (34) taken for the infrared region and the radio-frequency region coincide in full analogy with isotropic

systems. The longitudinal tensor of conductivity in the quasistatic limit is given by

$$\sigma_{ab}^{||}(\boldsymbol{k}, \omega) \cong -\frac{ie^2\omega}{k^2}\left(\frac{\partial N}{\partial \mu}\right)\left[1+\frac{\pi i}{2}\left(\frac{\omega}{\tilde{k}\tilde{V}}\right)\frac{1}{1+F_0}\right]\left(\frac{M^*}{m_a^*}\right)\delta_{ab}. \tag{35}$$

It is well-known that the real part of the function (34) renormalizes the frequencies of phonons in the "jellum" model whereas the imaginary part gives their lifetimes (see [13], Chapter 4). Since the real part of (34) remains unchanged in comparison to the isotropic case we have the same expression for phonon frequencies near $\boldsymbol{k} = 0$, *i. e.* the same expression for the phonon velocity. On the other hand, the imaginary part of the frequency (connected with the inverse lifetime) will now be anisotropic as a result of the anisotropy of the imaginary part of (34). It can be easily shown that the ratio of the imaginary and real parts of the phonon frequency is here multiplied by $(k/\tilde{k})$ in comparison to the result of [13].

In the quasihomogeneous limit (*i. e.* for $\omega \gg k\tilde{V}$) we have

$$\varepsilon(\boldsymbol{k}, \omega) \cong 1-\frac{4\pi e^2 N}{\omega^2}\left(\frac{1+F_1}{M^*}\right)\left(\frac{\tilde{k}}{k}\right)^2\left[1+\frac{3}{5}\left(\frac{\tilde{k}\tilde{V}}{\omega}\right)^2(1+F_1)\left(1+\frac{5}{9}F_0+\frac{4}{9}F_2\right)\right], \tag{36}$$

$$\sigma_{ab}^{||}(\boldsymbol{k}, \omega) \cong \frac{ie^2N(1+F_1)}{\omega m_a^*}A(\tilde{k}, \omega)\delta_{ab}, \tag{37}$$

where $A(\tilde{k}, \omega)$ denotes the expression in the square bracket of (36). As for isotropic systems, the expressions (36) and (37) differ from the analogous expressions for the infrared region; the inequalities proved in [4] are also fulfilled.

Let us consider the reaction to transversal fields. Since even $\varepsilon^{||}$ is not equal to $1+4\pi i\sigma^{||}/\omega$ there is no reason to introduce the transversal dielectric function, in contrast with the isotropic case. The reaction in the quasistatic limit, expressed in terms of the conductivity tensor, has the form

$$\sigma_{ab}^{\perp}(\boldsymbol{k}, \omega) \cong \frac{Nie^2(1+F_1)}{\omega m_a^*}\left[1+\frac{1}{5}\left(\frac{\tilde{k}\tilde{V}}{\omega}\right)^2(1+F_1)(1+F_2)\right]\delta_{ab} \tag{38}$$

and $\sigma^{\perp}$ coincides with $\sigma^{||}$ if $\boldsymbol{k}$ tends to zero, which has a simple physical meaning. Quite similarly to $\sigma^{||}$, (38) is not equal to the analogous quantity taken in the infrared region (*i. e.* for $m_a^* \to m_{ca}$ and $F_l \to A_l$). The inequalities analogous to those obtained in [4] will be also fulfilled here. We can analogously introduce the tensor of radio-frequency effective masses; its form in the symmetry axes reference frame is $[m_a^*/(1+F_l)]\,\delta_{ab}$. The elements of this tensor in comparison to optical masses m_a satisfy the same inequalities as for the isotropic systems. The response to the transversal field in the quasistatic limit is given by

$$\sigma_{ab}^{\perp}(\boldsymbol{p}, \omega) \cong \frac{3e^2N\pi}{4\tilde{k}u_F}\left[1+i\left(\frac{\omega}{\tilde{k}\tilde{V}}\right)R_2\right]\left(\frac{M^*}{m_a^*}\right)\delta_{ab}. \tag{39}$$

This formula can be obtained from our considerations [4] only in the limit $\omega \to 0$. Then we have the classical result describing the anomalous skin effect; here the result does not depend on the quasiparticle interaction and hence it is the same for the infrared and the radio-frequency regions. The small term proportional to R_2 is computed in Appendix B.

The term R_2 is always real and the correction to the main dissipative effect is dissipativeless. It should be emphasized that the correction term is rather important here, because the static transversal fields disappear. On the other hand, R_2 is presented by us as the infinite sum over l, where all amplitudes F_l appear. Hence,the application of the formula for R_2 is rather doubtful, with the possible exception of the additional verification of the sum formula (*cf.* [14]).

In our paper we have shown that the reaction of the considered system can be obtained from the reaction of the isotropic system provided that we replace the vector $\boldsymbol{k}$ by $\tilde{\boldsymbol{k}}$ in S^{ab}. Moreover, we shall perform the suitable transformation of the tensor S^{ab}. This transformation can be represented in terms of the current and field transformation. The same results were obtained recently in [8] by the application of the phenomenological approach. Taking into account that the microscopic and phenomenological approach are equivalent for isotropic and homogeneous systems [2] we find that the present approach and that developed in [8] are equivalent too. Our results for the infrared and radio-frequency regions coincide in a few lowest order terms of the quasistatic limit and differ in the quasihomogeneous limit. This is not a particular property of the considered systems, it is connected with the basic properties of the electron-phonon interaction.

The assumption of the square dispersion law of $\varepsilon(\boldsymbol{p})$ is our strongest assumption. It should be emphasized that even for a system with a spherical Fermi surface this is also a strong assumption (it is more natural to take $\varepsilon(\boldsymbol{p}) = f(\boldsymbol{p}^2)$, where f is some increasing function). For nonlinear functions f the Ward identity (5) is not fulfilled and the transformation (14) of the correlation functions S^{ab} for either a or $b > 0$ cannot be performed. In this case we ought to perform the analogous transformation with vertices in the "ω" limit replaced by the vertices in the "k" limit [1]. This allows us to express the correlation functions in terms of quasiparticle properties (*i. e.* by some integrals on the Fermi surface) at least in the infrared region. In the radio-frequency region some serious difficulties can appear and it is possible that the analogue of the additional transformation of correlation functions $I_{ab} \to \Phi_{ab} + I'_{ab}$ cannot be performed for neither a nor $b > 0$. Note that this difficulty does not have any influence on our statement that static properties with EPI taken into account and disregarded coincide. This is fulfilled because static properties are determined by the correlation function $\tilde{S}_{00}$ and the autocorrelation function for spin vertices. Moreover, the analogue of (5) is fulfilled for spin vertices with the accuracy up to spin-orbit coupling and the relativistic spin-spin interaction.

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APPENDIX A

Let us start from the formulae (24) and (25). Expressing there $\boldsymbol{E}'$ by $\boldsymbol{E}$ and ϱ we find that

$$\varrho = \frac{ie^2}{\omega} \sum_{b=1}^{3} E_b S^{0b}(\boldsymbol{k}, \omega) - \frac{4\pi e^2 \varrho}{\omega k^2} \sum_{b=1}^{3} k_b S^{0b}(\boldsymbol{k}, \omega), \tag{40}$$

$$J_a = \frac{ie^2}{\omega}\sum_{b=1}^{3} E_b S^{ab}(\boldsymbol{k},\omega) - \frac{4\pi e^2 \varrho}{\omega k^2}\sum_{b=1}^{3} k_b S^{ab}(\boldsymbol{k},\omega) + \\ + \frac{Ne^2 i}{m_a \omega} E_a - \frac{4\pi N e^2 \varrho}{m_a \omega k^2} k_a. \tag{41}$$

Taking into account the identities (22) and the formula (13) we find that the second sum in (40) is equal to $\omega \tilde{S}^{00}(\boldsymbol{k},\omega)/\varepsilon(\boldsymbol{k},\omega)$, where $\varepsilon(\boldsymbol{k},\omega)$ is defined by (32). Substituting this result into (40) and applying there the formula (13) once more we find (27). It can be also shown that the second sum of the formula (41) is equal to

$$-\frac{Nk_a}{m_a} + \frac{\omega S^{a0}(\boldsymbol{k},\omega)}{\varepsilon(\boldsymbol{k},\omega)}, \qquad a > 0. \tag{42}$$

In the proof of the formula (42) we have used the formulae (13,20) and (22) for $b > 0$. Now, replacing the second sum of the formula (41) by (42), expressing there ϱ from the formula (27) and applying (13) to S^{ab} under the first sum we see that (26) is fulfilled.

APPENDIX B

The transversal tensor of conductivity should be obtained from this quantity for isotropic systems by multiplying by the diagonal tensor $(M^*/m_a^*)\,\delta_{ab}$ and substituting $\tilde{k}$, $\tilde{V}$ instead of k, V (*cf.* Chapter 3). For the isotropic systems we have

$$S^{ab}(\boldsymbol{k},\omega) + \frac{N(1+F_1)}{M^*}\delta_{ab} = R'(k,\omega)\delta_{ab} + \hat{k}_a\hat{k}_b k^2 U(k,\omega), \tag{43}$$

where $\hat{k}_a = k_a/|\boldsymbol{k}|$. The transversal part of the above quantity can be represented by

$$R' = \frac{1}{2}\sum_{a=1}^{3}\sum_{b=1}^{3}\tilde{S}^{ab}[\delta_{ab} - \hat{k}_a\hat{k}_b] + \frac{N(1+F_1)}{M^*}. \tag{44}$$

In the quasistatic limit, with the accuracy suitable to obtain (39), the operator $Q(\boldsymbol{k},\omega)$ can be replaced by $-1 - L - L^2$, where L is equal to $(\omega/kV)\delta$ and $\delta = (\hat{\boldsymbol{k}}\hat{\boldsymbol{u}} - i\eta)^{-1}$, $\eta = 0^+$ The coefficients at $(\omega/kV)^n$ in (43) were obtained in [4] for $n = 0$ and 1. The coefficient for $n = 2$ is equal to

$$-\frac{3N(1+F_1)^2}{M^*}\langle \hat{\boldsymbol{u}}_a[\delta^2(1+F)^{-1} - (1+F)^{-2}F\delta^2 + \\ + (1+F)^{-2}F\delta(1+F)^{-1}F\delta - \delta(1+F)^{-2}F\delta]\hat{\boldsymbol{u}}_b\rangle, \tag{45}$$

where any algebraic expression containing F ought to be considered in the meaning of the algebra of operators. It should be noted that

$$C(F)\,(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}') = \sum_{l=0}^{\infty}(2l+1)C(F_l)P_l(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}'),$$

where C is an arbitrary rational function. On the other hand, such $C(F)$ which does not stand between deltas in (45) can be replaced by the numerical factor $C(F_1)$. Substituting (45) into (44), taking into account the above remarks and the equality $(\hat{\boldsymbol{u}}\hat{\boldsymbol{k}})\delta = 1$ we find that the coefficient at $(\omega/kV)^2$ in R' is equal to

$$-\frac{3N}{2M^*}\{\langle\delta^2\rangle - 1 + F_1\langle\delta[(1+F)^{-1}F]'\delta\rangle - F_1\langle[(1+F)^{-1}F]\rangle -$$

$$-(1+F_1)^2\langle\delta[(1+F)^{-2}F]'\delta\rangle + (1+F_1)^2\langle[(1+F)^{-2}F]\rangle\}, \tag{47}$$

where $C'(F)$ is defined as

$$C'(F)(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}') \equiv [C(F)(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}')]\cdot(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}'). \tag{48}$$

In the proof of the formula (47) we have used the identity

$$\sum_{a=1}^{3}\langle\hat{\boldsymbol{u}}_a\delta C(F)\delta\hat{\boldsymbol{u}}_a\rangle = \langle\delta C'(F)\delta\rangle. \tag{49}$$

Applying the recurrence formula for Legendre polynomials and the formulae (46), (48) we find that the l-th Legendre amplitude of the function $C'(F)$ is given by

$$C'_l = \frac{l}{2l+1}C(F_{l-1}) + \frac{l+1}{2l+1}C(F_{l+1}). \tag{50}$$

Taking into account the addition theorem for spherical functions, the definition of Legendre functions and (50) we can prove that $\langle C(F)\rangle = C(F_0)$ and

$$\langle\delta C'(F)\delta\rangle = \sum_{l=0}^{\infty}[lC(F_{l-1}) + (l+1)C(F_{l+1})][Q_l(i\eta)]^2. \tag{51}$$

Since $Q_l(i\eta) = Q_l(0) - (i\pi/2)P_l(0)$ thus, applying the formula (51) and the well-known formulae for Legendre functions and polynomials we obtain from (39), (47) that R_2 is given by

$$\frac{2}{\pi}\left\{1 + \frac{F_1F_0}{1+F_0} + \frac{F_0(1+F_1)^2}{(1+F_0)^2} - F_1\sum_{l=0}^{\infty}\left[\frac{lF_{l-1}}{1+F_{l-1}} + \frac{(l+1)F_{l+1}}{1+F_{l+1}}\right]\left[\frac{(l-1)!!}{l!!}\right]^2 g_l -\right.$$

$$\left. -(1+F_1)^2\sum_{l=0}^{\infty}\left[\frac{lF_{l-1}}{(1+F_{l-1})^2} + \frac{(l+1)F_{l+1}}{(1+F_{l+1})^2}\right]\left[\frac{(l-1)!!}{l!!}\right]^2 g_l\right\}. \tag{52}$$

Here g_l is defined as 1 for odd l and $-\pi^2/4$ for even l and we have assumed quite formally hat $(-1)!! = 0!! = 1$.

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