

# EQUIVALENCE RELATIONS FOR A LASER RESONATOR WITH A LENS-LIKE MEDIUM — APPROXIMATION RESULTING FROM THE HUYGENS-FRESNEL PRINCIPLE

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Using the Fresnel scalar formulation of the Huygens principle, the problem of determining the electromagnetic field distribution at the mirrors of an optical resonator filled with a medium displaying a parabolic variation of the index of refraction has been reduced to a simpler question of the so-called, passive equivalent resonator. The values of the equivalent resonator parameters are identical with those obtained by the Kogelnik approach, based on the ray matrix formulation of geometrical optics.

## 1. Introduction

Certain properties of the laser resonators filled with a lens-like medium as well the influence of such medium on the laser radiation have recently been considered in literature ([1], [2], [3], [4], [5], [6]). Some aspects of the forenamed resonators have been explained ([1], [5]) by the Kogelnik theory [7], based on the ray matrix approach of geometrical optics. Following the increasing interest in the resonators with lens-like media, an attempt is made in this paper to present the theory of such a resonator with a medium of which the index of refraction varies in radial direction (in the plane perpendicular to the resonator axis).

A parabolic radial variation of the index of refraction of a medium is assumed. Such an assumption comprises a broad variety of functions describing the actual distributions of the index of refraction. The system of integral equations, presented in this paper, describing the distribution of the electromagnetic field at the resonator mirrors, is obtained from the Fresnel scalar formulation of the Huygens principle ([8], [9]). This is permissible in cases where the dimensions of the mirrors are large in terms of wavelength, and when the field is very nearly a transverse electromagnetic one (the waves of TE and TM types) and it is uniformly polarized in one direction. Taking into consideration

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a relatively small change of the index of refraction of the medium and basing on the results of the work [10], it may be assumed that the application of the Huygens-Fresnel principle is justified in the cases in question.

## 2. System of integral equations and equivalence parameters of a resonator with lens-like medium

An optical resonator formed by two spherical (or flat) mirrors  $M_1$  and  $M_2$  spaced at a distance  $d$  is considered, as shown in Fig. 1. The radii of curvature of the mirrors are  $R_1$  and  $R_2$ , respectively. The mirrors are assumed to be circular, and of equal diame-

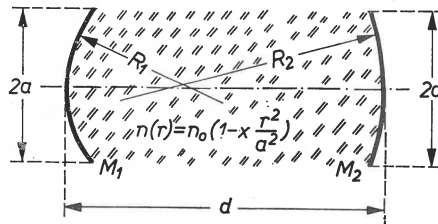


Fig. 1. Geometry of an optical resonator filled with a medium having a parabolic radial variation of the index of refraction

ters  $2a$ . All the resonator dimensions are assumed large against the wavelength. It is, additionally, assumed that

$$a \ll d, \quad (1)$$

which is typical for the optical resonators.

The medium filling all the space between the mirrors is assumed to be determined by the parabolic radial variation of the index of refraction

$$n(r) = n_0 \left( 1 - x \frac{r^2}{a^2} \right), \quad (2)$$

where:

$r$  — distance from the resonator axis,  $x$  — characteristic parameter of the medium.

The characteristic medium parameter  $x$  is negative ( $x < 0$ ) for the medium with defocusing properties and positive ( $x > 0$ ) for the focusing one. It is assumed that the characteristic medium parameter  $x$  is small against unity, *i.e.*,

$$|x| \ll 1. \quad (3)$$

Introducing the definition of the resonator mode in a steady state [9]:

$$\psi_a^{(j)} = [\gamma^{(j)}]^a \psi^{(j)}, \quad (j = 1 \text{ or } 2) \quad (4)$$

and using the Fresnel scalar formulation of the Huygens principle the system of integral equations describing the distribution of electromagnetic field at mirrors  $M_1$  and  $M_2$  of

the optical resonator with the medium having the index of refraction  $n(r)$  may be written down as follows:

$$\gamma^{(2)} \psi^{(2)}(r_2, \varphi_2) = \frac{ik}{2\pi} \int_0^a \int_0^{2\pi} \psi^{(1)}(r_1, \varphi_1) \frac{\exp[-ik \int_{M(r_1, \varphi_1, z_1)}^{N(r_2, \varphi_2, z_2)} n(r) ds]}{R} r_1 dr_1 d\varphi_1, \quad (5a)$$

$$\gamma^{(1)} \psi^{(1)}(r_1, \varphi_1) = \frac{ik}{2\pi} \int_0^a \int_0^{2\pi} \psi^{(2)}(r_2, \varphi_2) \frac{\exp[-ik \int_{N(r_2, \varphi_2, z_2)}^{M(r_1, \varphi_1, z_1)} n(r) ds]}{R} r_2 dr_2 d\varphi_2. \quad (5b)$$

It is seen that the introduced generalization of the Huygens-Fresnel principle for an inhomogeneous medium is confined in the exchange of geometrical distance  $kR$  between points  $M$  and  $N$  in the phases of the Huygens secondary disturbances by the extremal optical length  $\int n ds$ , resulting from the Fermat principle [8].

The resonator eigenfunction  $\psi^{(j)}(r_j, \varphi_j)$ , ( $j = 1$  or  $2$ ) in (5a, b) describes the distribution of the electromagnetic field at point  $(r_j, \varphi_j)$  on the  $j$ -th mirror (Fig. 2); the resonator

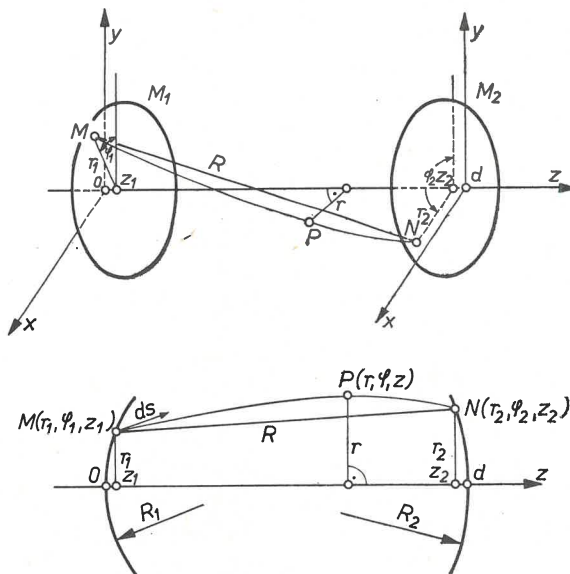


Fig. 2. Geometry of a passive optical resonator

eigenvalue  $\gamma^{(j)}$ , ( $j = 1$  or  $2$ ) corresponds to the eigenfunction  $\psi^{(j)}$  and specifies the diffraction losses and the phase shift to which the waves are exposed during each transit in the resonator, and  $k = \frac{\omega}{c}$  is the wave number.

Other symbols in (5a) and (5b) are defined in Fig. 1 and 2. By definition (4) it is meant that in a steady state the distribution of field at the  $j$ -th mirror reproduces itself within

the multiplicative constant  $\gamma^{(j)}$  after each transit, the number  $q$  specifies the successive transition of the wave in the resonator.

On account of the symmetry (the index exchange  $1 \rightleftharpoons 2$ ) of the both equations of system (5) only the first equation (5a) will be considered below.

To calculate the extremal optical path in a lens-like medium one uses the differential equation for light rays [8]. For paraxial rays this ray equation has the form:

$$n_0 \frac{d^2 r}{dz^2} = -2n_0 \frac{xr}{a^2} \quad (6)$$

for the distance  $r(z)$  of the optical path from the  $z$  axis. The solution of the differential equation (6) for the problem in question is

$$r = r_1 \cos \left[ \frac{\sqrt{2x}}{a} \left( z - \frac{r_1^2}{2R_1} \right) \right] + r_1' \frac{a}{\sqrt{2x}} \sin \left[ \frac{\sqrt{2x}}{a} \left( z - \frac{r_1^2}{2R_1} \right) \right], \quad (7)$$

where  $r_1$  and  $r_1'$  are the ray position and slope at the point  $M$  at the mirror  $M_1$ , respectively.

For paraxial rays  $\left( \left| \frac{dr}{dz} \right| \ll 1, \left| \frac{d\varphi}{dz} \right| \ll 1 \right)$ , the following approximation for a line element  $ds$  in a cylindrical coordinates  $r, \varphi, z$  is reasonable

$$ds = \left[ 1 + \frac{1}{2} \left( \frac{dr}{dz} \right)^2 + \frac{1}{2} r^2 \left( \frac{d\varphi}{dz} \right)^2 \right]^{1/2} dz. \quad (8)$$

Taking this into consideration, and assuming that

$$z_1 \cong \frac{r_1^2}{2R_1}, \quad (9)$$

$$z_2 \cong d - \frac{r_2^2}{2R_2}, \quad (10)$$

from simple but arduous calculations one obtains

$$\int_{M(r_1, \varphi_1, z_1)}^{N(r_2, \varphi_2, z_2)} n(r) ds = n_0 d + g_1^* \frac{r_1^2}{2d^*} + g_2^* \frac{r_2^2}{2d^*} - \frac{r_1 r_2 \cos(\varphi_2 - \varphi_1)}{d^*}, \quad (11)$$

where:

$$d^* = \frac{a \sin \left( \frac{\sqrt{2x}}{a} d \right)}{n_0 \sqrt{2x}}, \quad (12)$$

$$g_1^* = \cos \left( \frac{\sqrt{2x}}{a} d \right) - \frac{a}{R_1 \sqrt{2x}} \sin \left( \frac{\sqrt{2x}}{a} d \right), \quad (13)$$

$$g_2^* = \cos \left( \frac{\sqrt{2x}}{a} d \right) - \frac{a}{R_2 \sqrt{2x}} \sin \left( \frac{\sqrt{2x}}{a} d \right). \quad (14)$$

In calculation of the integral (11) the terms of third and higher orders with respect to  $r$  are neglected.

Assuming the approximation [9]  $R \cong d$  in the denominator of the kernel of integral equation (5a) and taking into account (11) and the expression [11]:

$$\begin{aligned} & \int_0^{2\pi} \exp \left[ ik \frac{r_1 r_2 \cos(\varphi_2 - \varphi_1)}{d^*} - im\varphi_1 \right] d\varphi_1 = \\ & = 2\pi \exp \left[ im \left( \frac{\pi}{2} - \varphi_2 \right) \right] I_m \left( \frac{kr_1 r_2}{d^*} \right), \end{aligned} \quad (15)$$

it can be seen after integrating (5a) with respect to  $\varphi_1$  that the function

$$\psi(r_1, \varphi_1) = \chi_m(r_1) \exp(-im\varphi_1), \quad (m = \text{integer}) \quad (16)$$

satisfies (5a).  $I_m$  in (15) is a Bessel function of the first kind and  $m$ -th order. Next, taking into consideration the symmetry relations (1  $\Leftrightarrow$  2) it can be directly shown that the functions  $\chi_m(r_1)$  and  $\chi_m(r_2)$  satisfy the reduced system of integral equations of the resonator with a medium of the index of refraction (2), expressed as:

$$\begin{aligned} \gamma^{(2)*} \chi_m^{(2)}(r_2) &= \frac{k}{d^*} \exp \left[ i(m+1) \frac{\pi}{2} - ikd^* \right] \int_0^a \chi_m^{(1)}(r_1) \exp \left[ -\frac{ik}{2d^*} (g_1^* r_1^2 + g_2^* r_2^2) \right] \times \\ & \times I_m \left( \frac{kr_1 r_2}{d^*} \right) r_1 dr_1, \end{aligned} \quad (17a)$$

$$\begin{aligned} \gamma^{(1)*} \chi_m^{(1)}(r_1) &= \frac{k}{d^*} \exp \left[ i(m+1) \frac{\pi}{2} - ikd^* \right] \int_0^a \chi_m^{(2)}(r_2) \exp \left[ -\frac{ik}{2d^*} (g_1^* r_1^2 + g_2^* r_2^2) \right] \times \\ & \times I_m \left( \frac{kr_1 r_2}{d^*} \right) r_2 dr_2, \end{aligned} \quad (17b)$$

where:

$$\gamma^{(j)*} = \frac{d}{d^*} \exp[-ik(d^* - n_0 d)] \gamma^{(j)}, \quad (j = 1 \text{ or } 2). \quad (18)$$

The system of integral equations (17) is the same as that for a passive optical resonator [12] with equivalent parameters given by (12)–(14), (18).

In this way the problem under consideration has been reduced to a simpler one, namely to the problem of the electromagnetic field distribution in a passive optical resonator.

The properties of the passive optical resonators with circular mirrors are described by the three generalized parameters [13]:

$$N = \frac{a_1 a_2 k}{2\pi d}, \quad (19)$$

$$G_1 = \frac{a_1}{a_2} g_1, \quad (20)$$

$$G_2 = \frac{a_2}{a_1} g_2, \quad (21)$$

where  $2a_1$  and  $2a_2$  are the diameters of the mirrors and parameters  $g_1$  and  $g_2$  have the form:

$$g_1 = 1 - \frac{d}{R_1}, \quad (22)$$

$$g_2 = 1 - \frac{d}{R_2}. \quad (23)$$

The parameters (19)–(21) of the resonator just considered are given by:

$$N^* = \frac{akn_0 \sqrt{x}}{\pi \sqrt{2} \sin\left(\frac{\sqrt{2x}}{a} d\right)}, \quad (24)$$

$$G_1^* = \cos\left(\frac{\sqrt{2x}}{a} d\right) - \frac{a}{R_1 \sqrt{2x}} \sin\left(\frac{\sqrt{2x}}{a} d\right), \quad (25)$$

$$G_2^* = \cos\left(\frac{\sqrt{2x}}{a} d\right) - \frac{a}{R_2 \sqrt{2x}} \sin\left(\frac{\sqrt{2x}}{a} d\right). \quad (26)$$

The passive optical resonator of parameters (24)–(26) or (12)–(14) may be referred to as a passive one equivalent to a resonator filled with a medium of the index of refraction described by expression (2). The form of parameters (24)–(26) of an equivalent resonator is identical with those obtained by Kogelnik's approach, based on the ray matrix formulation of geometrical optics.

### 3. Discussion

Expanding the expressions (24)–(26) in terms of the argument of trigonometric functions and neglecting the terms of third and higher orders it can be shown that for

$$|x| \ll \frac{a^2}{2d^2}, \quad (27)$$

which is satisfied for typical laser lens-like media (e.g., [1], [5]), the values of the equivalent resonator parameters  $N^*$ ,  $G_1^*$  and  $G_2^*$  (24)–(26) can be approximated as follows:

$$N^* = \frac{n_0(1+\alpha)ka^2}{2\pi d}, \quad (28)$$

$$G_1^* = g_1(1-\alpha) - 2\alpha, \quad (29)$$

$$G_2^* = g_2(1-\alpha) - 2\alpha, \quad (30)$$

where:

$$\alpha = \frac{xd^2}{3a^2}. \quad (31)$$

From the form (28)–(30) of the equivalent resonator parameters more simple than (24)–(26) one can draw almost immediately a number of practical conclusions. From expressions (29) and (30) it follows, for instance, that for the plane-parallel resonator ( $g_1 = g_2 = 1$ ) filled with a defocusing medium ( $x < 0$ ,  $\alpha < 0$ ; e.g. laser He Ne at  $\lambda = 6401 \text{ \AA}$ , [1]), the inequality  $G_1^*G_2^* > 1$  is satisfied; so that, according to the resonator stability condition ( $0 \leq G_1G_2 \leq 1$ , [13]), such a resonator is unstable. On the other hand, if the medium in the considered resonator is a focusing one ( $x > 0$ ,  $\alpha > 0$ ), such a system is always stable in cases of practical interest. It is seen, moreover, that a confocal resonator ( $g_1 = g_2 = 0$ ) is relatively hardly sensitive to the influence of both the focusing and defocusing media. Analogously one can derive the conclusions for the resonators of geometries other than the plane-parallel and confocal. As has been shown experimentally ([1], [5]) the lens-like medium has a very pronounced effect upon the resonator. For instance, the results of work [1] show that the resonator for the 6401  $\text{\AA}$  transition, consisting of a spherical ( $R = 215 \text{ cm}$ ) and flat mirror spaced at 217.9 cm, is equivalent to a similar passive resonator having a 222.2 cm spherical mirror and a mirror spacing of 215.34 cm. In the passive case the former resonator possesses high diffraction losses and is unstable ( $G_1G_2 < 0$ ) but the latter (equivalent) one occurs to be stable ( $0 \leq G_1G_2 \leq 1$ ). Hence it follows that for the 6401  $\text{\AA}$  oscillation the former resonator is a stable one. The results observed experimentally and those predicted by Kogelnik, as well as the present analysis are in good agreement.

It has also been experimentally proved [5] that the plasma of a  $\text{CO}_2$  laser exhibits a significant negative focusing defined by the plasma characteristics. The behaviour of the resonator filled with a plasma of this kind may also be predicted on the grounds of the theories discussed herewith.

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