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ON A RESOLUTION OF THE EQUATIONS GOVERNING THE SECOND ORDER CORRELATION FUNCTIONS FOR AN ISOTROPIC HYDROMAGNETIC TURBULENCE

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One presents the possibility of solving the system of Chandrasekhar's equations by using the power series expansions for the correlation functions.

1. Introduction

The statistical treatment of the theory of hydromagnetic turbulence involves an incomplete set of equations, the number of which is less than the number of unknown functions (*i. e.* of all sorts of correlations). Supplementary arguments are to be imposed in order to obtain a complete set of equations (see Millionstcikov [1]. Kraichnan [2]). Chandrasekhar [3] obtained, using the Millionstcikov hypotesis, a pair of equations for the second order correlation functions of the velocity and the magnetic field, respectively. These equations were solved by Chandrasekhar in the approximation $\nu = \lambda = 0$ and $\tau \to 0$.

The object of this paper is to give a formal solution to Chandrasekhar's equations for an isotropic, homogeneous and stationary turbulence by expanding the correlation functions in power series of independent variables. This method was proposed by Smirnov and Shapiro [4] in order to solve the equation for the second order correlation function of the velocity in the case of ordinary hydrodynamic turbulence.

2. The Resolution of Chandrasekhar's equations

Let us consider the following equations:

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial \tau^2} - \nu^2 D_5^2\right) Q = -2Q \frac{\partial}{\partial r} D_5 Q - 2H \frac{\partial}{\partial r} D_5 H \tag{1}$$

$$\left(\frac{\partial^2}{\partial \tau^2} - \lambda^2 D_5^2\right) H = -2Q D_5 H - 2H D_5 Q - 2 \frac{\partial Q}{\partial r} \frac{\partial H}{\partial r} \tag{2}$$

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where ν is the coefficient of kinematic viscosity; $\lambda = 1/4\pi\mu\sigma$, μ and σ are the coefficients of magnetic permeability and electrical conductivity, respectively. $Q(r, \tau)$ and $H(r\ \tau)$ are the scalar functions defined by the correlation tensors:

$$Q_{ij} = \langle u_i(\vec{r}', t')u_j(\vec{r}'', t'')\rangle$$

$$H_{ij} = \langle h_i(\vec{r}', t')h_j(\vec{r}'', t'')\rangle$$

Here, u_i and h_i are the components of the velocity and the components of the magnetic field divided by $(4\pi\varrho/\mu)^{1/2}$, $\vec{r}'$ and $\vec{r}''$ are the radius vectors of two neighbouring points, t', t'' are two instants of time, $r = |\vec{r}''-\vec{r}'|$; $\tau = t''-t'$, and the angular brackets denote ensemble averages. The operator $D_5 = \frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}$ is the five dimensional Laplacian operator.

For the sake of simplicity we shall consider the following longitudinal correlation functions $f(r, \tau)$ and $g(r, \tau)$:

$$f(r, \tau) = -\frac{2Q}{\langle u^2\rangle}; \qquad g(r, \tau) = -\frac{2H}{\langle h^2\rangle} \tag{3}$$

Substituting (3) into Eqs (1) and (2) we find:

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial\tau^2} - \nu^2 D_5^2\right)f = \langle u^2\rangle\left[f\frac{\partial}{\partial r}D_5 f + \alpha^2 g\frac{\partial}{\partial r}D_5 g\right] \tag{4}$$

$$\left(\frac{\partial^2}{\partial\tau^2} - \lambda^2 D_5^2\right)g = \langle u^2\rangle\left[fD_5 g + gD_5 f + \frac{\partial f}{\partial r}\frac{\partial g}{\partial r}\right] \tag{5}$$

where

$$\alpha = \langle h^2\rangle/\langle u^2\rangle$$

If the hydromagnetic turbulence is isotropic, homogeneous and stationary, $f(r, \tau)$ and $g(r, \tau)$ are even functions with respect to r and τ, and power series may be used for them:

$$f(r, \tau) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{i,k}r^{2i}\tau^{2k}(a_{0,0} = 1); \qquad g(r, \tau) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k}r^{2i}\tau^{2k}(b_{0,0} = 1) \tag{6}$$

We shall consider that these series are uniform convergent and they can be derived and integrated term by term. By substituting (6) into Eqs (4) and (5) we obtain:

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} ik(2k-1)a_{i,k}r^{2i-1}\tau^{2k-2} - 2\nu\sum_{i=3}^{\infty}\sum_{k=0}^{\infty} i(i-1)(i-2)(4i^2+8i+3)a_{i,k}r^{2i-5}\tau^{2k} =$$

$$= \langle u^2\rangle\left[\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=2}^{\infty}\sum_{m=0}^{\infty} l(l-1)(2l+3)(a_{i,k}a_{l,m}+\alpha^2 b_{i,k}b_{l,m})r^{2i+2l-3}\tau^{2k+2m}\right] \tag{7}$$

$$\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} k(2k-1)b_{i,k}r^{2i}\tau^{2k-2}-2\lambda^2\sum_{i=2}^{\infty}\sum_{k=0}^{\infty} i(i-1)(4i^2+8i+3)b_{i,k}r^{2i-4}\tau^{2k} =$$

$$= \langle u^2\rangle \Big[\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\sum_{m=0}^{\infty} l(2l+3)(a_{i,k}b_{l,m}+b_{i,k}a_{l,m})r^{2i+2l-2}\tau^{2k+2m}+$$

$$+2\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\sum_{m=0}^{\infty} il a_{i,k}b_{l,m}r^{2i+2l-2}\tau^{2k+2m}\Big] \tag{8}$$

Now, we equalize the coefficients corresponding to the same powers of r and τ, and we find:

$$a_{i+1,k+1} = \frac{1}{(i+1)(k+1)(2k+1)}[2v^2(i+1)(i+2)(i+3)(4i^2+32i+63)a_{i+3,k}+$$

$$+\langle u^2\rangle \sum_{p=0}^{i}\sum_{q=0}^{k}(p+1)(p+2)(2p+7)(a_{i-p,k-q}a_{p+2,q}+\alpha^2 b_{i-p,k-q}b_{p+2,q}] \tag{9}$$

$$b_{i,k+1} = \frac{1}{(k+1)(2k+1)}[2\lambda^2(i+1)(i+2)(4i^2+24i+35)b_{i+2,k}+$$

$$+\langle u^2\rangle \sum_{p=0}^{i}\sum_{q=0}^{k}(p+1)(2p+5)(a_{i-p,k-q}b_{p+1,q}+b_{i-p,k-q}a_{p+1,q})+$$

$$+2\langle u^2\rangle \sum_{p=0}^{i}\sum_{q=0}^{k}(i-p)(p+1)a_{i-p,k-q}b_{p+1,q}] \tag{10}$$

where $i, k \geqslant 0$

If the correlations decrease with the distance, it may by assumed that

$$\left(\frac{\partial^2}{\partial\tau^2}-v^2D_5^2\right)f(r,\tau)\Big|_{r=\infty} = 0 \tag{11}$$

and the integration of equation (4) with respect to r leads to:

$$\left(\frac{\partial^2}{\partial\tau^2}-v^2D_5^2\right)f(r,\tau) = -\langle u^2\rangle\Big[C(\tau)-\int_0^r f(r',\tau)\frac{\partial}{\partial r'}D_5 f(r',\tau)dr'+$$

$$+\alpha^2K(\tau)-\alpha^2\int_0^r g(r',\tau)\frac{\partial}{\partial r'}D_5 g(r',\tau)dr'\Big] \tag{12}$$

where we have considered:

$$C(\tau) = \int_0^\infty f(r',\tau)\frac{\partial}{\partial r'}D_5 f(r',\tau)dr' = \sum_{k=0}^{\infty} C_k\tau^{2k} \tag{13}$$

$$K(\tau) = \int_0^\infty g(r',\tau)\frac{\partial}{\partial r'}D_5 g(r',\tau)dr' = \sum_{k=0}^{\infty} K_k\tau^{2k} \tag{14}$$

By introducing (6) into Eq. (12) and identifying the coefficients of the terms with the same powers of r and τ, we find:

$$a_{0,k+1} = \frac{1}{2(k+1)(2k+1)} [280v^2 a_{2,k} - \langle u^2 \rangle (C_k + \alpha^2 K_k)] \tag{15}$$

For C_k and K_k it is easy to obtain from (13) and (14) the following expressions:

$$C_k = 4 \sum_{p=0}^{k} \int_0^\infty \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (l+1)(l+2)(2l+7) a_{i,k-p} a_{l+2,p} r^{2i+2l+1} dr \tag{16}$$

$$K_k = 4 \sum_{p=0}^{k} \int_0^\infty \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (l+1)(l+2)(2l+7) b_{i,k-p} b_{l+2,p} r^{2i+2l+1} dr \tag{17}$$

Now, from (9), (10) and (15) one concludes that if we know the coefficients $a_{i,0}$ and $b_{i,0}$, *i. e.* the correlation functions $f(r, 0)$ and $g(r, 0)$ (from the experiment or another theory), the functions $f(r, \tau)$ and $g(r, \tau)$ can be determined. One assumes also that $\langle u^2 \rangle$, $\langle h^2 \rangle$, v and λ are known.

3. The resolution of the generalized Chandrasekhar's equations

The previous procedure can be applyed to the more general equations, which involve terms describing the external force [5]:

$$-\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial \tau^2} - v^2 D_5^2\right) Q = 2Q \frac{\partial}{\partial r} D_5 Q + 2H \frac{\partial}{\partial r} D_5 H + \frac{\partial}{\partial r}\frac{\partial G_1}{\partial \tau} - v \frac{\partial}{\partial r} D_5 G_2 + \\ + \frac{\partial}{\partial r}\left(5 + r\frac{\partial}{\partial r}\right) I_2 \tag{18}$$

$$-\left(\frac{\partial^2}{\partial \tau^2} - \lambda^2 D_5^2\right) H = 2Q D_5 H + 2H D_5 Q + 2 \frac{\partial Q}{\partial r}\frac{\partial H}{\partial r} \tag{19}$$

where G_1 and G_2 represent the odd and even parts respectively with respect to the time τ of the scalar function G defined by the tensor $G_{ij} = \langle f_i', u_j'' \rangle$; f_i are the components of the external force per unit mass, and I_2 is the even part of the function I defined by the tensor $I_{ij,k} = \langle u_i' u_j' f_k'' \rangle$

Using the expressions (3), Eqs (18) and (19) become:

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial \tau^2} - v^2 D_5^2\right) f = \langle u^2 \rangle \left[f \frac{\partial}{\partial r} D_5 f + \alpha^2 g \frac{\partial}{\partial r} D_5 g \right] + \frac{2}{\langle u^2 \rangle}\frac{\partial}{\partial r} S \tag{20}$$

$$\left(\frac{\partial^2}{\partial \tau^2} - \lambda^2 D_5^2\right) g = \langle u^2 \rangle \left[f D_5 g + g D_5 f + \frac{\partial f}{\partial r}\frac{\partial g}{\partial r} \right] \tag{21}$$

where we have set:

$$S(r, \tau) = \frac{\partial G_1}{\partial \tau} - \nu D_5 G_2 + \left(r \frac{\partial}{\partial r} + 5 \right) I_2 \tag{22}$$

The function $S(r, \tau)$ is even with respect to r and τ and can be assumed as:

$$S(r, \tau) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} s_{i,k} r^{2i} \tau^{2k} \tag{23}$$

By using the previous procedure we obtain:

$$a_{i+1,k+1} = \frac{1}{(i+1)(k+1)(2k+1)} \Bigg[2\nu^2 (i+1)(i+2)(i+3)(4i^2+32i+63) a_{i+3,k} + \\ + \langle u^2 \rangle \sum_{p=0}^{i} \sum_{q=0}^{k} (p+1)(p+2)(2p+7)(a_{i-p,k-q} a_{p+2,q} + \\ + \alpha^2 b_{i-p,k-q} b_{p+2,q}) + \frac{(i+1)}{\langle u^2 \rangle} s_{i+1,k} \Bigg] \tag{24}$$

$$b_{i,k+1} = \frac{1}{(k+1)(2k+1)} [2\lambda^2 (i+1)(i+2)(4i^2+24i+35) b_{i+2,k} + \\ + \langle u^2 \rangle \sum_{p=0}^{i} \sum_{q=0}^{k} (p+1)(2p+5)(a_{i-p,k-q} b_{p+1,q} + b_{i-p,k-q} a_{p+1,q}) + \\ + 2\langle u^2 \rangle \sum_{p=0}^{i} \sum_{q=0}^{k} (i-p)(p+1) a_{i-p,k-q} b_{p+1,q}] \tag{25}$$

and

$$a_{0,k+1} = \frac{1}{2(k+1)(2k+1)} \left[280\nu^2 a_{2,k} - \langle u^2 \rangle (C_k + \alpha^2 K_k) + \frac{2}{\langle u^2 \rangle} s_{0,k} \right] \tag{26}$$

From these expressions results that the correlation functions $f(r, \tau)$ and $g(r, \tau)$ can be determined from the functions $f(r, 0)$, $g(r, 0)$ and from the correlation of the velocity with the external force. These results are a generalization of those derived by Smirnov [6] in the case of ordinary hydrodynamic turbulence.

4. Conclusions

The purpose of this paper has been to present the possibility of solving the system of Chandrasekhar's equations. If the condition (11) is fulfilled, this system of equations for $f(r, \tau)$, and $g(r, \tau)$ has the solutions in the class of functions expressed in power series (6). The solutions were obtained in the form of reccurrence formulae for the expansion coefficients. To determine these solutions it is necessary to know the coefficients $a_{i,0}$ and

$g_{i,0}$ (*i. e.* the correlation functions $f(r, 0)$ and $g(r, 0)$) from experiment or another theory of isotropic, homogeneous and stationary turbulence, which considers Millionstcikov's hypothesis as fulfilled. We mention that this hypothesis of quasinormality sometimes leads to unphysical conclusions [2] and it may be applied only for short time intervals.

The method used in this paper was proposed by Smirnov and Shapiro [4] for the hydrodynamic turbulence. This method was then generalized to include the case when external force is present.

Our paper giving a formal solution to Chandrasekhar's equations should arouse interest and attempts should be made to find the physical aspects of the results obtained and to verify them experimentally.

REFERENCES

[1] M. D. Millionstcikov, *Dokl. Akad. Nauk SSSR*, 32, 615 (1941).
[2] R. H. Kraichnan, *J. Fluid Mech.*, **5**, 497 (1959).
[3] S. Chandrasekhar, *Proc. Roy. Soc.*, **233A**, 322 (1955).
[4] V. I. Smirnov, B. S. Shapiro, *Trudy Centr. Aer. Obs.*, **78**, 80 (1967).
[5] S. Codreanu, *J. Phys. A.* (*Proc. Phys. Soc.*), **3**, 6, 150 (1970).
[6] V. I. Smirnov, *Izv. Nauk SSSR, Meh. Jid. Gaza*, **2**, 101 (1970).