ELECTROMAGNETIC WAVE DIFFRACTED AT SINGLE AND DOUBLE CRITICAL POINTS

By T. Niepokojczycki*

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Formulae for waves diffracted at double and single critical points are obtained by the stationary phase method on the basis of the electromagnetic Helmholtz-Huygens principle. The case of a spherical diverging wave and observation points far from the shadow boundary is considered.

Introduction

The electromagnetic theory of diffraction given by Kottler (1923) provides and approximate solution to the diffraction problem for the Kirchhoff screen. Kottler, following the liines set by Larmor, properly interpreted the electromagnetic Helmholtz-Huygens principle and built a theory analogical to the Kirchhoff theory of diffraction for the scalar case, especially emphasizing the point of view of Young's type of interpretation. The application of the stationary phase method to Kottler's formulae makes it possible to present the electric and magnetic field strengths, $E^{(D)}$ and $H^{(D)}$, of the diffracted wave in a way which is simple and easy to interpret from the physical point of view.

Karczewski and Petykiewicz (1967) used the method of stationary phase in the Kottler theory for obtaining a formula for the diffracted wave originating at single critical points, valid for any region of observation.

In this work the stationary phase method is applied to Kottler formulae obtained in the approximate form for large wave numbers k and observation regions far from the shadow boundary. Approximate expressions for the wave diffracted at single and double critical points, valid for points of observation distant from the shadow boundary, are obtained. These formula, in the case of single critical points, are identical within the range of applicability with the formulae of the above-cited authors. Also, the polarization state of the single contributions to the diffracted wave is discussed.

1. *Kottler's theory*

The Kirchhoff theory of diffraction in the scalar case makes use of Huygens' principle and certain assumptions, the so-called Kirchhoff conditions, as to the state of the field on the screen and diffracting aperture.

* Address: Łódź, Czysta 29, Poland.

Kirchhoff defines the state of the field at any point of observation P within a region R by means of a certain function satisfying the vibrational equation. This function depends on the point Q lying on the surface F bounding the region R. Kottler extends this procedure to the vector functions of a electromagnetic field, *viz.* $E(Q)$ and $H(Q)$.

Assuming that Sommerfeld's conditions are fulfilled at infinity, Kottler gets the following expression for the electric field $E(P)$ and magnetic field $H(P)$ at the observation point P (Kottler 1923):

$$\boldsymbol{E}(P) = \boldsymbol{E}_k + \frac{1}{4\pi} \int_B (d\boldsymbol{s} \times \boldsymbol{E}_0(Q))\omega + \frac{1}{4\pi i k} \operatorname{grad}_P \int_B (d\boldsymbol{s} \cdot \boldsymbol{H}_0(Q))\omega \qquad (1.1a)$$

$$\boldsymbol{H}(P) = \boldsymbol{E}_k + \frac{1}{4\pi} \int_B (d\boldsymbol{s} \times \boldsymbol{H}_0(Q))\omega - \frac{1}{4\pi i k} \operatorname{grad}_P \int_B (d\boldsymbol{s} \cdot \boldsymbol{E}_0(Q))\omega. \qquad (1.1b)$$

The vector components

$$\boldsymbol{E}_k = \boldsymbol{E}_0(L, P) + \frac{1}{4\pi} \int_S df \left\{ \boldsymbol{E}_0(Q) \frac{\partial}{\partial n} \omega - \omega \frac{\partial}{\partial n} \boldsymbol{E}_0(Q) \right\}. \qquad (1.2a)$$

$$\boldsymbol{H}_k = \boldsymbol{H}_0(L, P) + \frac{1}{4\pi} \int_S df \left\{ \boldsymbol{H}_0(Q) \frac{\partial}{\partial n} \omega - \omega \frac{\partial}{\partial n} \boldsymbol{H}_0(Q) \right\} \qquad (1.2b)$$

are an analogue of Kirchhoff's solutions for the scalar case. The surface integrals extend over the illuminated part of the screen S, and n denotes the normal to S directed towards the shadow half-space. $\boldsymbol{E}_0(L, P)$ and $\boldsymbol{H}_0(L, P)$ represent the field of the incident geometrical wave from the source L at point P. The function $\omega = e^{ikr}/r$ is the spherical-symmetrical solution of the vibrational equation, r standing for the distance between the observation point P and point of integration Q. $\boldsymbol{E}_0(Q)$ and $\boldsymbol{H}_0(Q)$ are the respective values of incident electric and magnetic fields at point Q. The edge integrals are found from the jump values of the tangential components of the electromagnetic field at the edge B of aperture f in the screen. These discontinuities of the electromagnetic field's tangential components at the screen edge are equal to the values $\boldsymbol{E}_0(Q)$ and $\boldsymbol{H}_0(Q)$ of the incident wave's field at the given point Q of the edge B. The subscript P at the operation of differentiation means differentiation is done with respect to the coordinates of the observation point P.

The addition of the edge integrals has the purpose of correcting the surface integrals so that the whole would satisfy Maxwell's equations. When the surface F is closed and the values of electromagnetic field at F are continuous, the integrals vanish and the Kottler representation of Huygens' principle transforms into the scalar Huygens principle for the various components of the electromagnetic field vectors.

As the source of the field undergoing diffraction Kottler takes a vibrating linear electric dipole, whose moment is parallel to an arbitrarily oriented constant unit vector. The field of the spherical wave emitted by the dipole has at point Q the form

$$\boldsymbol{E}_0(Q) = \frac{1}{4\pi}\,\mathrm{grad}_L \frac{\partial}{\partial t_L}\frac{e^{ik\varrho}}{\varrho} + \frac{k^2}{4\pi}\boldsymbol{t}\frac{e^{ik\varrho}}{\varrho}. \tag{1.3a}$$

$$\boldsymbol{H}_0(Q) = -\frac{ik}{4\pi}\left(\boldsymbol{t}\times\mathrm{grad}_L\frac{e^{ik\varrho}}{\varrho}\right). \tag{1.3b}$$

The subscript L at the gradient operation denotes differentiation with respect to the coordinates of the source L; $\varrho = |\boldsymbol{LQ}|$ is the distance between source L and point of integration Q.

The solutions $\boldsymbol{E}_k$ and $\boldsymbol{H}_k$ are solutions of the jump problem, analogous to Kirchhoff's solution in the scalar case.

The subsequent procedure with the $\boldsymbol{E}_k$ and $\boldsymbol{H}_k$ vector components is identical with that in the scalar theory. In the case of an incident spherical wave ((1.3a) and (1.3b)) we get the following expressions for the state of the field in the shadow half-space

$$\boldsymbol{E}_k = \eta\boldsymbol{E}_0 + \frac{1}{4\pi}\left(\mathrm{grad}_L\frac{\partial}{\partial t_L} + \boldsymbol{t}k^2\right)u^{(D)}(L, P) \tag{1.4a}$$

$$\boldsymbol{H}_k = \eta\boldsymbol{H}_0 + \frac{ik}{4\pi}(\boldsymbol{t}\times\mathrm{grad}_L\, u^{(D)}(L, P)). \tag{1.4b}$$

η denotes the Heaveside function. In the light cone $\eta = 1$, whereas beyond it $\eta = 0$. The characteristic function

$$u^{(D)} = \frac{1}{4\pi}\int_B \frac{e^{ik\zeta}}{r\varrho}\frac{(\boldsymbol{r}\times\boldsymbol{\varrho})}{r\varrho+\boldsymbol{r\varrho}}\cdot\boldsymbol{ds} \tag{1.5}$$

appearing in Eqs (1.4a) and (1.4b) is the well-known diffracted wave for the scalar case. By ζ the phase function $\zeta = r+\varrho$ is denoted.

Finally, after replacing the terms $\boldsymbol{E}_k$ in (1.1a) and $\boldsymbol{H}_k$ in (1.1b) by Eqs (1.4a) and (1.4b), respectively, the solution to the diffraction problem for a spherical incident wave takes the form (Kottler 1923)

$$\boldsymbol{E}(P) = \eta\boldsymbol{E}_0 + \frac{1}{4\pi}\left(\mathrm{grad}_L\frac{\partial}{\partial t_L} + \boldsymbol{t}k^2\right)u^{(D)}(L, P) + \frac{1}{4\pi}\int_B(\boldsymbol{ds}\times\boldsymbol{E}_0(Q))\omega +$$
$$+ \frac{1}{4\pi ik}\,\mathrm{grad}_P\int_B(\boldsymbol{ds}\cdot\boldsymbol{H}_0(Q))\omega \tag{1.6a}$$

$$\boldsymbol{H}(P) = \eta\boldsymbol{H}_0 + \frac{ik}{4\pi}(\boldsymbol{t}\times\mathrm{grad}_L\, u^{(D)}(L, P)) + \frac{1}{4\pi}\int_B(\boldsymbol{ds}\times\boldsymbol{H}_0(Q))\omega +$$
$$- \frac{1}{4\pi ik}\,\mathrm{grad}_P\int_B(\boldsymbol{ds}\cdot\boldsymbol{E}_0(Q))\omega. \tag{1.6b}$$

The electromagnetic field represented by the edge integrals depends, in the case of a spherical incident wave, only on the state of the field at the points Q of the edge. Thus, Eqs (1.6a) and (1.6b) are a mathematical formulation of Young's ideas. According to them, the state of the electromagnetic field at the observation point P is the result of interference between the directly incident wave and the wave reflected by the aperture edge. According with this approach, the expressions

$$\boldsymbol{E}^{(D)} = \frac{1}{4\pi}\left(\text{grad}_L \frac{\partial}{\partial t_L} + \boldsymbol{t}k^2\right) u^{(D)} + \frac{1}{4\pi} \int_B \omega(\boldsymbol{ds} \times \boldsymbol{E}_0(Q)) +$$

$$+ \frac{1}{4\pi ik} \text{grad}_P \int_B \omega(\boldsymbol{ds} \cdot \boldsymbol{H}_0(Q)) \tag{1.7a}$$

$$\boldsymbol{H}^{(D)} = -\frac{ik}{4\pi} (\text{grad}_L\, u^{(D)} \times \boldsymbol{t}) + \frac{1}{4\pi} \int_B \omega(\boldsymbol{ds} \times \boldsymbol{H}_0(Q) +$$

$$- \frac{1}{4\pi ik} \text{grad}_P \int_B \omega(\boldsymbol{ds} \cdot \boldsymbol{E}_0(Q)) \tag{1.7b}$$

are defined as the diffracted electromagnetic wave.

2. Approximate form of Kottler's formulae for observation regions distant from the shadow boundary

Before applying the stationary phase method to Kottler's formulae, we shall approximate them as follows. We consider an electromagnetic field of such a high frequency that all terms proportional to the wave number k at a power lower than that of the k in the leading term may be neglected in Eqs (1.6a) and (1.6b).

In observation regions placed far from the shadow boundary all terms in Kottler's formulae behave regularly. Bearing this in mind, we make the following assumptions (Appendix 1):

$$\text{grad}_L \frac{\partial}{\partial t_L} u^{(D)} \sim -\frac{k^2}{4\pi} \int_B f(s) e^{ik\zeta} \nabla_L \varrho \frac{\partial}{\partial t_L} \varrho ds \tag{2.1}$$

$$\text{grad}_P \int_B (\boldsymbol{ds} \cdot \boldsymbol{H}_0) \frac{e^{ikr}}{r} \sim -\frac{k^2}{4\pi} ik \int_B \frac{e^{ik\zeta}}{r\varrho} [(\boldsymbol{t} \times \nabla_L \varrho) \cdot \boldsymbol{s}_0] \nabla_P r ds \tag{2.2}$$

$$\text{grad}_L\, u^{(D)} \sim \frac{ik}{4\pi} \int_B f(\;) e^{ik\zeta} \nabla_L \varrho ds \tag{2.3}$$

$$\text{grad}_P \int_B (\boldsymbol{s}_0 \cdot \boldsymbol{E}_0) \frac{e^{ikr}}{r} ds \sim -\frac{k^2}{4\pi} ik \int_B \frac{e^{ik\zeta}}{r\varrho} \left\{ \boldsymbol{s}_0 \left[\boldsymbol{t} + \frac{\partial \varrho}{\partial t} (\nabla_L \varrho) \right] \right\} \nabla_P r ds \tag{2.4}$$

$$\operatorname{grad}_L \frac{\partial}{\partial t_L} \frac{e^{ik\varrho}}{\varrho} \sim -\frac{k^2}{\varrho} e^{ik\varrho} \frac{\partial \varrho}{\partial t_L} \nabla_L \varrho \tag{2.5}$$

$$\operatorname{grad}_L \frac{e^{ik\varrho}}{\varrho} \sim ik \frac{e^{ik\varrho}}{\varrho} \nabla_L \varrho \tag{2.6}$$

where

$$f(s) = \frac{1}{4\pi} \frac{1}{r\varrho} \frac{(\boldsymbol{r}\times\boldsymbol{\varrho})\cdot\boldsymbol{s}_0}{r\varrho+\boldsymbol{r}\boldsymbol{\varrho}}. \tag{2.7}$$

$\boldsymbol{s}_0$ denotes the versor of the tangent to the element $\boldsymbol{ds}$.

We now introduce a set of three vectors $\boldsymbol{e}, \boldsymbol{h}$ and $\boldsymbol{\varrho}$, respectively defining the direction of the electric and magnetic field and the direction of the incident wave's propagation.

$$\boldsymbol{e} = \boldsymbol{t} - \frac{\partial \varrho}{\partial t} \nabla_L \varrho = \boldsymbol{\varrho}_0 \times (\boldsymbol{t} \times \boldsymbol{\varrho}_0) \tag{2.8}$$

$$\boldsymbol{h} = \boldsymbol{t} \times \boldsymbol{\varrho}_0 \tag{2.9}$$

$$\boldsymbol{\varrho}_0 = \nabla_Q \varrho. \tag{2.10}$$

After putting (2.5) into (1.3a) and (2.6) into (1.3b) we get with the help of vectors $\boldsymbol{e}$ and $\boldsymbol{h}$ the formulae for the field of the incident wave at point Q,

$$\boldsymbol{E}_0(Q) \sim \frac{k^2}{4\pi} \frac{e^{ik\varrho}}{\varrho} \boldsymbol{e} \tag{2.11a}$$

$$\boldsymbol{H}_0(Q) \sim \frac{k^2}{4\pi} \frac{e^{ik\varrho}}{\varrho} \boldsymbol{h}. \tag{2.11b}$$

When Eq. (2.11a) is substituted into (1.7a) and correspondingly (2.11b) into (1.7b) the diffracted wave, owing to (2.1), (2.2) and (2.4), is described in terms of the set of vectors $\boldsymbol{e}, \boldsymbol{h}$ and $\boldsymbol{\varrho}$ as follows:

$$\boldsymbol{E}^{(D)} \sim \frac{k^2}{16\pi^2} \int_B \frac{e^{ik\zeta}}{r\varrho} \left[\frac{(\boldsymbol{r}\times\boldsymbol{\varrho})\cdot\boldsymbol{s}_0}{r\varrho+\boldsymbol{r}\boldsymbol{\varrho}} \boldsymbol{e} + (\boldsymbol{s}_0\times\boldsymbol{e}) + (\boldsymbol{h}\cdot\boldsymbol{s}_0)\boldsymbol{r}_0 \right] ds \tag{2.12a}$$

$$\boldsymbol{H}^{(D)} \sim \frac{k^2}{16\pi^2} \int_B \frac{e^{ik\zeta}}{r\varrho} \left[\frac{(\boldsymbol{r}\times\boldsymbol{\varrho})\cdot\boldsymbol{s}_0}{r\varrho+\boldsymbol{r}\boldsymbol{\varrho}} \boldsymbol{h} + (\boldsymbol{s}_0\times\boldsymbol{h}) + (\boldsymbol{e}\cdot\boldsymbol{s}_0)\boldsymbol{r}_0 \right] ds. \tag{2.12b}$$

We denote by

$$\boldsymbol{g}_E = \frac{(\boldsymbol{r}\times\boldsymbol{\varrho})\cdot\boldsymbol{s}_0}{r\varrho+\boldsymbol{r}\boldsymbol{\varrho}} \boldsymbol{e} + (\boldsymbol{s}_0\times\boldsymbol{e}) + (\boldsymbol{h}\cdot\boldsymbol{s}_0)\boldsymbol{r}_0 \tag{2.13a}$$

and

$$\boldsymbol{g}_H = \frac{(\boldsymbol{r}\times\boldsymbol{\varrho})\cdot\boldsymbol{s}_0}{r\varrho+\boldsymbol{r}\boldsymbol{\varrho}} \boldsymbol{h} + (\boldsymbol{s}_0\times\boldsymbol{h}) + (\boldsymbol{e}\cdot\boldsymbol{s}_0)\boldsymbol{r}_0 \tag{2.13b}$$

the direction of the electric and magnetic fields, respectively, for an elementary diffracted wave originating at element ds.

Owing to the approximations (2.11a), (2.11b), (2.12a) and (2.12b), and with the aid of Eqs (2.13a) and (2.13b), Kottler's formulae (1.6a) and (1.6b) take on the simple form for observation regions placed far from the shadow boundary:

$$\boldsymbol{E}(P) \sim \frac{k^2}{4\pi}\left[\eta \frac{e^{ikR}}{R}\boldsymbol{e}_P + \frac{1}{4\pi}\int_B \frac{e^{ik\zeta}}{r\varrho}\, g_E ds\right] \tag{2.14a}$$

$$\boldsymbol{H}(P) \sim \frac{k^2}{4\pi}\left[\eta \frac{e^{ikR}}{R}\boldsymbol{h}_P + \frac{1}{4\pi}\int_B \frac{e^{ik\zeta}}{r\varrho}\, g_H ds\right] \tag{2.14b}$$

where

$$\boldsymbol{e}_P = \boldsymbol{R}_0 \times (\boldsymbol{t} \times \boldsymbol{R}_0)$$

$$\boldsymbol{h}_P = \boldsymbol{t} \times \boldsymbol{R}_0$$

$$\boldsymbol{R}_0 = \frac{\boldsymbol{LP}}{|\boldsymbol{LP}|} = \frac{\boldsymbol{R}}{R}.$$

The form of Eqs (2.14a) and (2.14b) is convenient in the application of the stationary phase method for finding asymptotic expressions describing the diffracted wave.

3. *Calculation of the diffracted wave by the stationary phase method*

If we assume a finite number ν of single critical points Q_j and a finite number μ of double critical points Q_{j*}, the field of the diffracted wave at the point of observation is given by

$$\boldsymbol{E}^{(D)} = \sum_{j=1}^{\nu} \boldsymbol{E}_j^{(D)} + \sum_{j*=1}^{\mu} \boldsymbol{E}_{j*}^{(D)} \tag{3.1a}$$

$$\boldsymbol{H}^{(D)} = \sum_{j=1}^{\nu} \boldsymbol{H}_j^{(D)} + \sum_{j*=1}^{\mu} \boldsymbol{H}_{j*}^{(D)} \tag{3.1b}$$

where $\boldsymbol{E}_j^{(D)}$ and $\boldsymbol{H}_j^{(D)}$ are contributions from Q_j, and
$\boldsymbol{E}_{j*}^{(D)}$ and $\boldsymbol{H}_{j*}^{(D)}$ are contributions from Q_{j*}.

We obtain the expression for the contribution $\boldsymbol{E}_j^{(D)}$ and $\boldsymbol{H}_j^{(D)}$ to the diffracted wave coming from a single critical point (by applying the stationary phase method directly to the formulae (2.14a) and (2.14b)) in the form

$$\boldsymbol{E}_j^{(D)} \sim \sqrt{\frac{2\pi}{|\zeta''|}}\, \frac{k^{3/2}}{16\pi^2}\, \frac{e^{i\left(k\zeta_j \pm \frac{\pi}{4}\right)}}{r\varrho}\, g_E\Big|_j \tag{3.2a}$$

$$\boldsymbol{H}_j^{(D)} \sim \sqrt{\frac{2\pi}{|\zeta''|}}\, \frac{k^{3/2}}{16\pi^2}\, \frac{e^{i\left(k\zeta_j \pm \frac{\pi}{4}\right)}}{r\varrho}\, g_H\Big|_j . \tag{3.2b}$$

Expressions in the same form as above can be obtained for the diffracted wave in observation regions far from the shadow boundary from the general formulae of Karczewski and Petykiewicz (1967) if we make use of the approximations given by these authors and utilize the approximation given in Section 2 of this work.

It follows from the shape of the functions g_E (2.13a) and g_H (2.13b) that the applicability of Eqs (3.2a) and (3.2b) is limited. At the shadow boundary these expressions become infinite, whereas the diffracted wave actually only undergoes a jump by a value equal to the incident wave, as was shown in the electromagnetic case by Karczewski (1961).

A further restriction on the applicability of the formulae written above stems from the assumption that in the expansion of the phase function $\zeta(s)$ into a series the first two non-vanishing terms are sufficient to represent the function $e^{ik\zeta}$ in the vicinity of the critical point with adequate accuracy. This assumption requires the ζ_j'' to be adequately large; for small ζ_j'' formulae (3.2a) and (3.2b) cannot be applied. In such a case subsequent terms of the expansion of the function $\zeta(s)$ must be taken into consideration. In particular, when the active regions of two critical points overlap account must be taken of the existence of two extreme values of the function $\zeta(s)$. In mathematical terms this corresponds to the expansion

$$\zeta(s) = \zeta_j + \frac{1}{2}(s-s_j)^2\zeta_j'' + \frac{1}{6}(s-s_j)^3\zeta_j'''. \tag{3.3}$$

Using this expansion we find the contribution to the diffracted wave (2.13a) and (2.13b) originating at a double critical point, *viz.*,

$$E_{j*}^{(D)} = \frac{k^2}{16\pi^2}\frac{e^{ik\zeta_j}}{r\varrho}\, g_{E_j} \int\limits_{s_j-\Delta s_j}^{s_j+\Delta s_j} e^{ik\left[\frac{1}{2}(s-s_j)^2\zeta_j'' + \frac{1}{6}(s-s_j)^3\zeta_j'''\right]} ds \tag{3.4a}$$

$$H_{j*}^{(D)} = \frac{k^2}{16\pi^2}\frac{e^{ik\zeta_j}}{r\varrho}\, g_{H_j} \int\limits_{s_j-\Delta s_j}^{s_j+\Delta s_j} e^{ik\left[\frac{1}{2}(s-s_j)^2\zeta_j'' + \frac{1}{6}(s-s_j)^3\zeta_j'''\right]} ds. \tag{3.4b}$$

With the use of a new variable w defined by the equation

$$s-s_j = \sqrt[3]{\frac{3\pi}{k\zeta_j'''}}\, w - \frac{\zeta_j''}{\zeta_j'''}$$

the integral over s may be written in the form (Rubinowicz 1924a)

$$\int\limits_{s_j-\Delta s_j}^{s_j+\Delta s_j} e^{ik\left[\frac{1}{2}(s-s_j)^2\zeta_j'' + \frac{1}{6}(s-s_j)^3\zeta_j'''\right]} ds$$

$$= \sqrt[3]{\frac{2\pi}{k\zeta_j'''}}\, e^{ik\frac{\zeta_j''^3}{3\zeta_j'''^2}} \int\limits_{\sqrt[3]{\frac{k\zeta_j'''}{3\pi}}\left(-\Delta s_j + \frac{\zeta_j''}{\zeta_j'''}\right)}^{\sqrt[3]{\frac{k\zeta_j''}{3\pi}}\left(\Delta s_j + \frac{\zeta_j''}{\zeta_j'''}\right)} e^{i\frac{\pi}{2}(w^3 - mw)}\, dw. \tag{3.5}$$

The condition $\Delta s_j > \left| \frac{\zeta_j''}{\zeta_j'''} \right|$ on the active element of the arc enables us at $k \to \infty$ to extend the limits of integration from $-\infty$ to $+\infty$ and resolve the integral (3.4b) into a double Airy integral. After applying the above transformation to the integrals in Eqs (3.4a) and (3.4b) we get the contribution of the double critical point to the diffracted wave,

$$\boldsymbol{E}_{j*}^{(D)} \sim \frac{k^2}{8\pi^2} \sqrt[3]{\frac{3\pi}{k\zeta_j'''}} \frac{e^{ik\left(\zeta_j + \frac{\zeta_j''^3}{3\zeta_j'''^3}\right)}}{r\varrho} \boldsymbol{g}_{Ej} Ai(m) \tag{3.6a}$$

$$\boldsymbol{H}_{j*}^{(D)} \sim \frac{k^2}{8\pi} \sqrt[3]{\frac{3\pi}{k\zeta_j'''}} \frac{e^{ik\left(\zeta_j + \frac{\zeta_j''^3}{3\zeta_j'''^2}\right)}}{r\varrho} \boldsymbol{g}_{Hj} Ai(m) \tag{3.6b}$$

where

$$Ai(m) = \int_0^{+\infty} \cos\left(\frac{\pi}{2} w^3 - mw\right) dw$$

$$m = \sqrt[3]{\frac{3k^2}{\pi^2}} \frac{\zeta_j''^2}{\zeta_j'''^{4/3}}.$$

The diffracted wave (3.1a) and (3.1b) calculated by means of (3.2a), (3.2b) and (3.6a), (3.6b) has the form

$$\boldsymbol{E}^{(D)} \sim \sum_{j=1}^{\nu} A_j \boldsymbol{g}_{Ej} + \sum_{j*=1}^{\mu} A_{j*} Ai(m)\, \boldsymbol{g}_{Ej} \tag{3.7a}$$

$$\boldsymbol{H}^{(D)} \sim \sum_{j=1}^{\nu} A_j \boldsymbol{g}_{Hj} + \sum_{j*=1}^{\mu} A_{j*} Ai(m)\, \boldsymbol{g}_{Hj} \tag{3.7b}$$

where

$$A_j = \frac{k^{3/2}}{16\pi^2} \sqrt{\frac{2\pi}{|\zeta_j''|}} \frac{e^{i\left(k\zeta_j \pm \frac{\pi}{4}\right)}}{r\varrho}$$

$$A_{j*} = \frac{k^2}{8\pi^2} \sqrt[3]{\frac{3\pi}{k\zeta_j'''}} \frac{e^{ik\left(\zeta_j + \frac{\zeta_j''^3}{3\zeta_j'''^2}\right)}}{r\varrho}.$$

4. Structure and polarization of contributions to the diffracted wave coming from an active element ds of the edge

When the formulae (3.4a) and (3.4b) for the contribution of an element ds of the diffracting edge to the diffracted wave are derived it is assumed that the element ds contains an active point Q_j at which the phase function takes on extreme values. This condition leads to the equality (Rubinowicz 1924a, 1966)

$$\cos(\varrho, ds) = -\cos(r, ds).$$

The senses of the vectors ϱ and r are towards the element ds.

This result means in the first approximation that the element ds reflects the incident wave (2.11a) and (2.11b) over the surface of a cone with the point Q_j as its vertex, an angle of the axial section equal to $2\times\sphericalangle(\varrho, ds)$ and axis determined by the vector $\boldsymbol{ds}$.

The contribution to the diffracted wave is linearly polarized. The plane of vibrations is determined by the vectors $-\boldsymbol{r}$ and $\boldsymbol{g}_{E_j}$, where $-\boldsymbol{r}$ denotes the direction of wave propagation. By $\boldsymbol{g}_{E_j}$ we denote the vector of electric field vibrations for the diffracted wave contribution coming from the active point Q_j,

$$\boldsymbol{g}_{Ej} = \boldsymbol{q}(r)+\boldsymbol{w}_j+\boldsymbol{r}_0 \cos(h, ds) \tag{4.1}$$

where

$$\boldsymbol{q}(r) = \frac{\cos(n, r)}{1+\cos(r, \varrho)} \sin(\varrho, ds)\boldsymbol{e} \tag{4.1a}$$

and

$$\boldsymbol{w}_j = \boldsymbol{s}_0\times\boldsymbol{e}$$

n denotes the external normal to the shadow boundary at the active point of the element ds. Vectors $\boldsymbol{w}_j$ and $\boldsymbol{e}$ define the plane π_1 on which the vector $\boldsymbol{u}_j = \boldsymbol{w}_j+\boldsymbol{q}_j(r)$ lies. The vector $\boldsymbol{g}_{Ej}$ lies in the plane π_2 determined by the vectors $\boldsymbol{r}_j$ and $\boldsymbol{u}_j$ (Fig. 1).

The plane π_2 is a plane intersecting the cone Q_j along a pair of generating lines of the cone, of which one is the direction of the incident ray, $-\boldsymbol{r}$. At fixed parameters $\boldsymbol{s}_0$, $\boldsymbol{e}$ and $\boldsymbol{\varrho}$ the direction of the vector $\boldsymbol{g}_{E_j}$ depends only on the direction of observation PQ_j. The po-

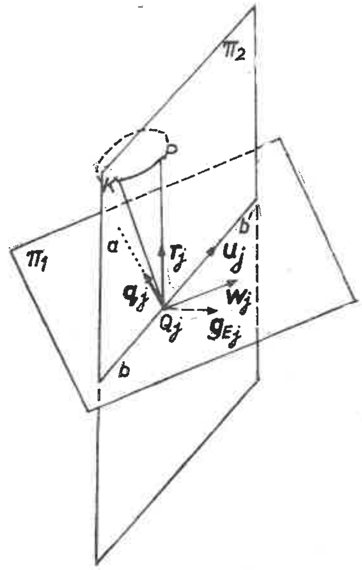

Fig. 1. Graphical presentation of the plane of vibrations π_2 of the vector $\boldsymbol{g}_{E_j}$. a — trace of element ds on plane π_1, bb' — line of intersection of planes π_1 and π_2, P — point of observation, $\sphericalangle KQ_jP$ — the angle arising due to the intersection of the cone Q_j by the plane π_2

sition of the plane π_1 is fixed; the plane π_2 intersects the cone variously, depending on the position of the observation point.

In a peculiar case, when the cone is degenerated into a plane ($\boldsymbol{\varrho} \perp d\boldsymbol{s}$ and $\boldsymbol{e} \parallel \boldsymbol{s}_0$), Eq. (4.1) becomes

$$\boldsymbol{g}_{E_j} = \frac{\cos(n, r)}{1+\cos(r, \varrho)}\,\boldsymbol{e}. \tag{4.2}$$

The directions of the vibrations of vectors of the incident wave and the contribution to the diffracted wave become parallel. In general, the directions of the vibrations of the vector $\boldsymbol{e}$ and vector $\boldsymbol{g}_{E_j}$ make a certain angle φ defined by

$$\cos\varphi = \frac{\boldsymbol{e}\cdot\boldsymbol{g}_{E_j}}{|\boldsymbol{e}|\,|\boldsymbol{g}_E|}$$

$$= \frac{q(r)+(\boldsymbol{h}\cdot\boldsymbol{s}_0)(\boldsymbol{r}\cdot\boldsymbol{e})}{\sqrt{q^2(r)+(\boldsymbol{w}\cdot\boldsymbol{h})^2+(\boldsymbol{w}\cdot\boldsymbol{\varrho}_0)^2+(\boldsymbol{h}\cdot\boldsymbol{s}_0)^2+2\boldsymbol{h}\cdot\boldsymbol{s}_0[q(r)\cos(r, e)+\boldsymbol{w}\cdot\boldsymbol{h}\cos(r, h)+\boldsymbol{w}\cdot\boldsymbol{\varrho}_0\cos(r, \varrho)]}} \tag{4.3}$$

where $q(r)$ is given by Eq. (4.1a).

In the case when the vector $\boldsymbol{e}$ lies in the plane determined by the vectors ϱ and ds, we have the relations $\boldsymbol{h} \perp \boldsymbol{s}_0$, $w\cdot\varrho = 0$ and $\boldsymbol{h} \parallel \boldsymbol{w}$, thereby considerably simplifying Eq. (4.3). We then have

$$\cos\varphi|_j = \left.\frac{\sin(\varrho, ds)\cos(n, r)}{\sqrt{\sin^2(\varrho, ds)\cos^2(n, r)+\sin^2(ds, e)\cos^2(w, h)(1+\cos(r, \varrho))}}\right|_j. \tag{4.4}$$

The formulae given here are unsuitable for calculating φ near the shadow boundary. The approximations (2.1) and (2.3) are invalid in this region.

The polarization of the contribution to the diffracted wave near the shadow boundary can be determined according to approximate formulae derived by Karczewski and Petykiewicz (1967).

We give here the simpler formulation of the approximate expression for the vector $\boldsymbol{E}_j^{(D)}$ near the shadow boundary, namely,

$$\boldsymbol{E}_j^{(D)} \sim -\frac{k^2 e^{ikR}}{8\pi R}\,\boldsymbol{g}_{E_j}$$

where

$$\boldsymbol{g}_{E_j} = \boldsymbol{e}\cos\alpha - iR\sqrt{\frac{1}{2\pi k\,|\zeta_j''|}}\,\frac{e^{i\frac{\pi}{4}}}{r\varrho}\,(\boldsymbol{s}_0\cdot\boldsymbol{h})\boldsymbol{r}.$$

The value of $\cos\alpha$ tends to ± 1, the sign depending on whether we approach the boundary of shadow from the side of the light cone or from the side of the geometrical shadow. The parallelism of the vectors $\boldsymbol{e}$ and $\boldsymbol{g}_{E_j}$ and the large value of the diffracted wave's amplitude give rise to very distinct interference fringes. When the point of observation on the cone surface at Q_j withdraws from the shadow boundary, the direction of the vector $\boldsymbol{r}$ deviates from the direction ϱ, and this causes a rotation of the plane π_2 and increase in $\sphericalangle(g_{E_j}, e)$.

As a consequence, only the parallel components of the vectors $\boldsymbol{E}(\boldsymbol{P})$ and $\boldsymbol{E}_j^{(D)}(P)$ interact, what must bear some effect on the weakening of the distinctness of the Fresnel fringes.[1]

When the element ds contains a double critical point, the Eqs (3.6a) and (3.6b) apply. In them, the Airy integral appears as the interference term. Interference is caused by the interaction between the diffracted waves originating at two closely lying extrema, Q^j and Q_{j*} (Rubinowicz 1924a). The interaction takes place along the image curve Z_j, that is, the parabola which arises due to the intersection of the cones with vertices Q_j and Q_{j*}. The curve Z_j is placed symmetrically on both sides of the element ds, in the plane perpendicular to the strictly tangential plane.

When the point of observation moves along a fixed cone having its vertex at point Q_j, the direction of $\boldsymbol{g}_{E_j}$ is always the same function of r, regardless of whether the point P lies in the range of applicability of Eqs (3.4a) and (3.4b) or Eqs (3.6a) and (3.6b). The presence of the other extremum does not affect the position of the polarization plane.

The state of polarization of the diffracted wave (3.7a) and (3.7b) is determined by the superposition of the polarization states coming from the various active elements of the edge B.

The polarization of the diffracted wave in the Fraunhofer region has been examined by Karczewski and Wolf (1966) by means o coherence matrices. As a result they found that if the incident wave is linearly polarized, then the diffracted wave is also polarized linearly.

I wish to express my appreciation to Professor W. Rubinowicz for having proposed the topic of this work.

APPENDIX A

We give here the calculations which lead to the approximations accepted in Section 2.

a) Accomplishment of the operation grad $\frac{\partial}{\partial t_L}$ on the function $u^{(D)}$ (1.5) leads to the equality

$$\operatorname{grad}_L \frac{\partial}{\partial t_L} u^{(L)} = \frac{1}{4\pi} \int_B \operatorname{grad}_L \frac{\partial}{\partial t_L} e^{ik\zeta} f(s) ds.$$

After operations under the integration sign are carried out, we have

$$\operatorname{grad}_L \frac{\partial}{\partial t_L} u^{(D)} = \frac{1}{4\pi} \int_B \operatorname{grad}_L e^{ik\zeta} \left(ikf(s) \frac{\partial \varrho}{\partial t_L} + \frac{\partial}{\partial t_L} f(s) \right) ds$$

$$= \frac{1}{4\pi} \int_B \left[ik \left(ikf(s) \frac{\partial \varrho}{\partial t_L} + \frac{\partial}{\partial t_L} f(s) \right) e^{ik\zeta} \nabla_L \varrho + e^{ik\zeta} \operatorname{grad}_L \left(ikf(s) \frac{\partial \varrho}{\partial t_L} + \right.\right.$$

$$\left.\left. + \frac{\partial}{\partial t_L} f(s) \right) \right] ds.$$

[1] A distinct drop in the contrast of fringes when withdrawing from the shadow boundary may be observed in the experiment devised by Nienhuis (1948) (see Rubinowicz 1966, Abb. 26).

Once the factor $e^{ik\zeta}$ is taken out before the bracket, we get inside the bracket a series which is asymptotic with respect to powers of k, namely,

$$\operatorname{grad}_L \frac{\partial}{\partial t_L} u^{(D)} = -\frac{k^2}{4\pi} \int_B \left[f(s) \frac{\partial \varrho}{\partial t_L} \nabla_L \varrho + \frac{1}{ik} \left(\nabla_L \varrho \frac{\partial}{\partial t_L} f(s) + \operatorname{grad}_L f(s) \frac{\partial \varrho}{\partial t_L} \right) - \frac{1}{k^2} \operatorname{grad} \frac{\partial}{\partial t_L} f(s) \right] e^{ik\zeta} ds.$$

Bearing in mind the assumptions made in Section 2, we limit ourselves to the first term and get (2.1).

b) Calculation of $\operatorname{grad}_P \int_B (\boldsymbol{s}_0 \cdot \boldsymbol{E}_0) \frac{e^{ikr}}{r} ds$, where E_0 is given by (1.3a)

We find that

$$\operatorname{grad}_L \frac{\partial}{\partial t_L} \frac{e^{ik\varrho}}{\varrho} \sim -k^2 \frac{e^{ik(\varrho)}}{\varrho} \frac{\partial \varrho}{\partial t_L} \nabla_L \varrho. \qquad \text{(A1)}$$

Putting (A1) into (1.3a) we have

$$\boldsymbol{E}_0 \sim -\frac{k^2}{4\pi} \frac{e^{ik\varrho}}{\varrho} \left(\frac{\partial \varrho}{\partial t_L} \nabla_L \varrho + \boldsymbol{t} \right). \qquad \text{(A2)}$$

By virtue of (A2) we get

$$\operatorname{grad}_P \int_B (\boldsymbol{s}_0 \cdot \boldsymbol{E}_0) \frac{e^{ikr}}{r} ds \sim -\frac{k^2}{4\pi} \int_B e^{ik\zeta} \left\{ \frac{ik}{\varrho} \left[\left(\frac{\partial \varrho}{\partial t_L} \nabla_L \varrho + \boldsymbol{t} \right) \cdot \boldsymbol{s}_0 \right] \nabla_p r + \operatorname{grad}_P \left[\frac{1}{r\varrho} \left(\frac{\partial \varrho}{\partial t} \nabla_L \varrho + \boldsymbol{t} \right) \boldsymbol{s}_0 \right] \right\} ds.$$

Proceeding in the same manner as in a), we restrict ourselves to the first term and thus get (2.4)

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