MICROSCOPIC THEORY OF DILUTE He^3—He II SOLUTIONS. I. DERIVATION OF HYDRODYNAMIC EQUATIONS WITHOUT VISCOUS TERMS

By Z. M. Galasiewicz

Institute of Theoretical Physics, University of Wrocław*
Institute of Low Temperatures and Structure Research, Polish Academy of Sciences

(Received November 24, 1970)

The Khalatnikov hydrodynamic equations for dilute He^3 — He II solutions are obtained on the basis of microscopic theory proposed by Bogolyubov.

Introduction

In the paper by Hohenberg and Martin [1] the development of the theory of liquid helium is divided into three levels. The first one begins with the unique macroscopic aspects of the superfluid state. To this level belongs, among others, the paper by Landau [2], giving the two-fluid hydrodynamic equations.

The second level is named semiphenomenological. This level is based on the theory of Landau [2], [3] which treates liquid helium as a gas of weakly interacting elementary excitations: phonons and rotons.

The third level is a totally microscopic one. Very important contributions to this level are papers by Bogolyubov [4] and Hohenberg and Martin [1] published in 1964–65 and based on similar ideas.

In 1952 Khalatnikov [5] obtained hydrodynamic equations for dilute solutions of He^3 in He II. This was accomplished, according to the Hohenberg and Martin classification, on the first macroscopic level. The aim of the present paper is to give the derivation of these equations in the schema proposed by Bogolyubov [4], *i.e.* on the microscopic level.

1. *Preliminary identities*

Consider a system composed of Bose and Fermi particles. The Bose field operators are denoted by $\psi(t, r)$ and $\psi^+(t, r)$, and Fermi field operators by $\tilde{\psi}(t, x)$ and $\tilde{\psi}^+(t, x)$, with $x = r, s$, where s is a spin index.

* Address: Instytut Fizyki Teoretycznej, Uniwersytet Wrocławski, Wrocław, Cybulskiego 36, Poland.

The Hamiltonian of our system has the form

$$\hat{H} = \hat{H}_B + \hat{H}_F + \hat{H}^{\text{int}}_{BF} + \delta H_t, \tag{1}$$

where

$$\hat{H}_B = \frac{1}{2m_B} \int \nabla\psi^+(t, r)\nabla\psi(t, r)d\boldsymbol{r} - \lambda^B \int \hat{\varrho}_B(t, r)d\boldsymbol{r} + \\ + \frac{1}{2} \iint V(r-r')\psi^+(t, r)\hat{\varrho}_B(t, r')\psi(t, r)d\boldsymbol{r}d\boldsymbol{r}', \tag{2}$$

$$\hat{H}_F = \frac{1}{2m_F} \sum_s \int \nabla\tilde{\psi}^+(t, x)\nabla\tilde{\psi}(t, x)d\boldsymbol{r} - \lambda^F \int \hat{\varrho}_F(t, r)d\boldsymbol{r} + \\ + \frac{1}{2} \iint V(r-r') \sum_s \tilde{\psi}^+(t, x)\hat{\varrho}_F(t, r')\tilde{\psi}(t, x)d\boldsymbol{r}d\boldsymbol{r}' \tag{3}$$

$$\hat{H}^{\text{int}}_{BF} = \frac{1}{2} \iint V(r-r') \, [\hat{\varrho}_B(t, r)\hat{\varrho}_F(t, r') + \hat{\varrho}_B(t, r')\hat{\varrho}_F(t, r)]d\boldsymbol{r}d\boldsymbol{r}', \tag{4}$$

$$\delta\hat{H}_t = \int \{\eta(t, r)\psi^+(t, r) + \eta^*(t, r)\psi(t, r)\}d\boldsymbol{r} + \\ + \int U(t, r)\hat{\varrho}(t, r)d\boldsymbol{r}, \tag{5}$$

$$\hat{\varrho}_B(t, r) = \psi^+(t, r)\psi(t, r), \quad \hat{\varrho}_F(t, r) = \sum_s \tilde{\psi}^+(t, x)\tilde{\psi}(t, x), \\ \hat{\varrho}(t, r) = \hat{\varrho}_B(t, r) + \hat{\varrho}_F(t, r) \tag{6}$$

here λ_B and λ_F are constants, $U(t, r)$ is an external scalar potential, and $\eta(t, r)$ and $\eta^*(t, r)$ are so called "sources of particles" [4]. They are given time and space dependent functions. We write r instead of $\boldsymbol{r}$ in the argument of the functional dependence and put $\hbar = 1$.

The assumption about the relatively small number of the Fermi particles enables us to consider interactions only between the He^4—He^4 and He^3—He^3 atoms, represented by the same function $V(r-r')$ [6]. Now we take also into account the interactions between the He^3—He^3 atoms described, too, by the function $V(r-r')$ (see [7]).

The Bose and Fermi field operators obey the following commutations relations

$$\psi(t, r)\psi^+(t, r') - \psi^+(t, r')\psi(t, r) = \delta(r-r'), \tag{7}$$

$$\tilde{\psi}(t, x)\tilde{\psi}^+(t, x') + \tilde{\psi}^+(t, x')\tilde{\psi}(t, x) = \delta(x-x'). \tag{8}$$

Now we are interested in the time derivatives of the following local quantities

$$\varrho^m(t, r) = \langle m_B\hat{\varrho}_B(t, r) + m_F\hat{\varrho}_F(t, r)\rangle = \langle\hat{\varrho}^m\rangle, \tag{9}$$

$$\boldsymbol{j}(t, r) = \langle\hat{\boldsymbol{j}}_B(t, r) + \hat{\boldsymbol{j}}_F(t, r)\rangle = \boldsymbol{j}_B(t, r) + \boldsymbol{j}_F(t, r),$$

$$\boldsymbol{j}_B(t, r) = \frac{i}{2} \langle(\nabla\psi^+(t, r))\psi(t, r) - \psi^+(t, r)\,(\nabla\psi(t, r))\rangle, \tag{10}$$

$$\boldsymbol{j}_F(t, r) = \frac{i}{2} \sum_s \langle (\nabla \tilde{\psi}^+(t, x)) \tilde{\psi}(t, x) - \tilde{\psi}^+(t, x) (\nabla \tilde{\psi}(t, x)) \rangle,$$

$$\begin{aligned} \varrho^m(t, r)\varepsilon(t, r) = & -\frac{1}{4m_B} \langle (\nabla^2 \psi^+(t, r))\psi(t, r) + \psi^+(t, r) (\nabla^2 \psi(t, r)) \rangle + \\ & + \frac{1}{2} \int V(r-r') \langle \psi^+(t, r) \hat{\varrho}_B(t, r') \psi(t, r) \rangle d\boldsymbol{r}' - \\ & - \frac{1}{4m_F} \sum_s \langle (\nabla^2 \tilde{\psi}^+(t, x)) \tilde{\psi}(t, x) + \tilde{\psi}^+(t, x) (\nabla^2 \tilde{\psi}(t, x)) \rangle + \\ & + \frac{1}{2} \int V(r-r') \sum_s \langle \tilde{\psi}^+(t, x) \hat{\varrho}_F(t, r') \tilde{\psi}(t, x) \rangle d\boldsymbol{r}' + \\ & + \frac{1}{2} \int V(r-r') \langle \hat{\varrho}_B(t, r) \hat{\varrho}_F(t, r') + \hat{\varrho}_B(t, r') \hat{\varrho}_F(t, r) \rangle d\boldsymbol{r}', \end{aligned} \tag{11}$$

$$\Phi(t, r) = \langle \psi(t, r) \rangle, \qquad \Phi^*(t, r) = \langle \psi^+(t, r) \rangle, \tag{12}$$

$$c(t, r) = \frac{\overset{m}{\varrho}_F(t, r)}{\varrho^m(t, r)}, \qquad \overset{m}{\varrho}_F = \langle m_F \hat{\varrho}_F \rangle = c(t, r) \varrho^m(t, r), \tag{13}$$

ϱ^m denotes the mean mass density of the solution, $\boldsymbol{j}$ the mean current, ε the energy per unit mass, and c the mass concentration of Fermi liquid (He^3). The brackets $\langle \ldots \rangle$ denote the nonequilibrium expectation values of the field operators.

In the paper of Bogolyubov [4] the mean density of particles, and the mean energy, entropy, free energy and chemical potential per particle were considered. Now we consider a mixture of particles with two different masses m_B and m_F. Hence, we must introduce the mean mass density and thermodynamic functions per unit mass. We have now also the additional local quantity, namely, the mass concentration $c(t, r)$ of He^3.

In further considerations we shall describe the nonequilibrium expectation values in terms of the following thermodynamic parameters: ϱ^m, θ, $\boldsymbol{v}_s$, $\boldsymbol{v}_n$ and c, where θ is temperature, $\boldsymbol{v}_s$ velocity of the superfluid and $\boldsymbol{v}_n$ of the normal component. Use is primarily made of the pair of parameters (P, θ) where P is pressure. Later, we shall demostrate how to express the suitable thermodynamic derivatives depending on (ϱ^m, θ) in terms od (P, θ).

For calculation of the time derivatives of the quantities defined by (9)–(13) we use the formula

$$i\dot{\hat{A}} = [\hat{A}, \hat{H}]. \tag{14}$$

The basic equations are

$$\begin{aligned} i\frac{\partial \psi(t, r)}{\partial t} &= -\lambda^B \psi(t, r) - \frac{\nabla^2 \psi(t, r)}{2m_B} + \int V(r-r') \hat{\varrho}(t, r') d\boldsymbol{r}' \psi(t, r) + U(t, r)\psi(t, r) + \eta(t, r), \\ i\frac{\partial \psi^+(t, r)}{\partial t} &= \lambda^B \psi^+(t, r) + \frac{\nabla^2 \psi^+(t, r)}{2m_B} - \psi^+(t, r) \int V(r-r') \hat{\varrho}(t, r') d\boldsymbol{r}' - U(t, r)\psi^+(t, r) - \eta^*(t, r), \end{aligned} \tag{15}$$

$$i\,\frac{\partial\tilde{\psi}(t,x)}{\partial t}=-\lambda^F\tilde{\psi}(t,x)-\frac{\nabla^2\tilde{\psi}(t,x)}{2m_F}+\int V(r-r')\hat{\varrho}(t,r')d\boldsymbol{r}'\tilde{\psi}(t,x)+U(t,r)\tilde{\psi}(t,x),$$

$$i\,\frac{\partial\tilde{\psi}^+(t,x)}{\partial t}=\lambda^F\tilde{\psi}^+(t,x)+\frac{\nabla^2\tilde{\psi}^+(t,x)}{2m_F}-\tilde{\psi}^+(t,x)\int V(r-r')\hat{\varrho}(t,r')d\boldsymbol{r}'-U(t,r)\tilde{\psi}^+(t,x).$$

With the help of (15) and (11) we have the following equations:

$$\frac{\partial\varrho^m(t,r)}{\partial t}+\nabla\boldsymbol{j}(t,r)=im_B[\eta^*(t,r)\Phi(t,r)-\eta(t,r)\Phi^*(t,r)],\tag{16}$$

$$\frac{\partial\varrho_F^m(t,r)}{\partial t}+\nabla\boldsymbol{j}_F(t,r)=0\rightarrow\frac{\partial c(t,r)\varrho^m(t,r)}{\partial t}+\nabla j_F(t,r)=0,\tag{17}$$

$$\begin{aligned}\frac{\partial j_\alpha(t,r)}{\partial t}&=\frac{1}{4}\left[\frac{1}{m_B}\nabla^2\varrho_B(t,r)+\frac{1}{m_F}\nabla^2\varrho_F(t,r)\right]-\\&-\sum_s\frac{\partial}{\partial r_\beta}\left[\frac{1}{m_B}\left\langle\frac{\partial\psi^+(t,r)}{\partial r}\frac{\partial\psi(t,r)}{\partial r_\beta}+\frac{\partial\psi^+(t,r)}{\partial r_\beta}\frac{\partial\psi(t,r)}{\partial r_\alpha}\right\rangle+\right.\\&\left.+\sum_s\frac{1}{m_F}\left\langle\frac{\partial\tilde{\psi}^+(t,x)}{\partial r_\alpha}\frac{\partial\tilde{\psi}(t,x)}{\partial r_\beta}+\frac{\partial\tilde{\psi}^+(t,x)}{\partial r_\beta}\frac{\partial\tilde{\psi}(t,x)}{\partial r_\alpha}\right\rangle\right]-\\&-\frac{1}{2}\int\frac{\partial V(R)}{\partial R_\alpha}\{\tilde{\mathscr{D}}_t(r,-R)+\tilde{\mathscr{D}}_t(r-R,R)\}d\boldsymbol{R}+\end{aligned}\tag{18}$$

$$+\varrho_B\frac{\partial}{\partial r_\alpha}(\lambda^B-U)+\varrho_F\frac{\partial}{\partial r_\alpha}(\lambda^F-U)+\frac{1}{2}\left(\frac{\partial\Phi^*}{\partial r_\alpha}\eta+\frac{\partial\Phi}{\partial r_\alpha}\eta^*-\Phi^*\frac{\partial\eta}{\partial r_\alpha}-\Phi\frac{\partial\eta^*}{\partial r_\alpha}\right),$$

$$\tilde{\mathscr{D}}_t(r,r'-r)=\mathscr{D}_t(r,r'-r)+\mathscr{D}_t^{BF}(r,r'-r)=\tilde{\mathscr{D}}_t(r,-R)$$

$$=\tilde{\mathscr{D}}_t(r-R,R)$$

$$\mathscr{D}_t(r,r'-r)=\langle\psi^+(t,r)\hat{\varrho}_B(t,r')\psi(t,r)\rangle+$$

$$+\sum_s\langle\tilde{\psi}^+(t,x)\hat{\varrho}_F(t,r')\tilde{\psi}(t,x)\rangle=\mathscr{D}_t(r,-R)=\mathscr{D}_t(r-R,R),$$

$$\mathscr{D}_t^{BF}(r,r'-r)=\langle\hat{\varrho}_B(t,r)\hat{\varrho}_F(t,r')+\hat{\varrho}_B(t,r')\hat{\varrho}_F(t,r)\rangle,$$

$$\boldsymbol{R}=\boldsymbol{r}-\boldsymbol{r}'$$

and

$$\frac{\partial\varrho^m\varepsilon}{\partial t}=\frac{i}{8}\nabla^2\left\{\frac{1}{m_B^2}\langle(\nabla^2\psi^+)\psi-\psi^+(\nabla^2\psi)\rangle+\sum_s\frac{1}{m_F^2}\langle(\nabla^2\tilde{\psi}^+)\tilde{\psi}-\tilde{\psi}^+(\nabla^2\tilde{\psi})\rangle\right\}+$$

$$+\sum_\beta\frac{\partial}{\partial r_\beta}\left\{\frac{i}{4}\left[\frac{1}{m_B^2}\left\langle\frac{\partial\psi^+}{\partial r_\beta}(\nabla^2\psi)-(\nabla^2\psi^+)\frac{\partial\psi}{\partial r_\beta}\right\rangle+\right.\right.$$

$$+\sum_s \frac{1}{m_F^2}\left\langle \frac{\partial \tilde{\psi}^+}{\partial r_\beta}(\nabla^2\tilde{\psi})-(\nabla^2\tilde{\psi}^+)\frac{\partial\tilde{\psi}}{\partial r_\beta}\right\rangle +$$

$$+\int V(R)\tilde{G}_t^{(\beta)}(r, R)d\boldsymbol{R}\Big\} + \sum_\beta \int \frac{\partial V(R)}{\partial R_\beta}[\tilde{G}_t^{(\beta)}(r, -R)+\tilde{G}_t^{(\beta)}(r-R, R)]d\boldsymbol{R}-$$

$$-\frac{1}{m_B}\sum_\beta j_B^{(\beta)}\frac{\partial}{\partial r_\beta}(U-\lambda^B)-\frac{1}{m_F}\sum_\beta j_F^{(\beta)}\frac{\partial}{\partial r_\beta}(U-\lambda^F)+$$

$$+\frac{i}{4m_B}(\eta\nabla^2\Phi^*-\eta^*\nabla^2\Phi+\Phi^*\nabla^2\eta-\Phi\nabla^2\eta^*)+ \tag{19}$$

$$+\frac{i}{2}\int V(r-r')\,[\eta^*(t,r)\langle\hat{\varrho}(t,r')\psi(t,r)\rangle+\eta^*(t,r')\langle\hat{\varrho}(t,r)\psi(t,r')\rangle-$$

$$-\eta(t,r')\langle\psi^+(t,r')\hat{\varrho}(t,r)\rangle-\eta(t,r)\langle\psi^+(t,r)\hat{\varrho}(t,r')\rangle]d\boldsymbol{r}',$$

$$\tilde{G}_t^{(\alpha)}(r, r'-r) = G_t^{(\alpha)}(r, r'-r)+G_t^{BF(\alpha)}(r, r'-r) = \tilde{G}_t^{(\alpha)}(r, -R) = \tilde{G}_t^{(\alpha)}(r-R, R),$$

$$\tilde{G}_t^{(\alpha)} = -\frac{1}{2m_B}\langle\psi^+(t,r)\hat{j}_B^{(\alpha)}(t,r')\psi(t,r)\rangle-\frac{1}{2m_F}\sum_s\langle\tilde{\psi}^+(t,x)\hat{j}_F^{(\alpha)}(t,r)\tilde{\psi}(t,x)\rangle,$$

$$G_t^{BF(\alpha)}(r, -R) = -\left\langle\frac{1}{2m_B}\hat{j}_B^{(\alpha)}(t,r)\hat{\varrho}_F(t,r')+\frac{1}{2m_F}\hat{j}_F^{(\alpha)}(t,r)\hat{\varrho}_B(t,r')\right\rangle.$$

The equations for $\Phi(t, r) = \langle\psi(t, r)\rangle = a \exp[i\chi(t, r)]$ and for $\Phi^*(t, r) = \langle\psi^+(t, r)\rangle$ are obtained after averaging the expressions (15) for $\partial\psi^+/\partial t$. We have no Fermi-pairs condensate; therefore, the mean values $\langle\psi\psi\rangle$ do not appear in $\delta\hat{H}_t$.

2. *Derivation of the hydrodynamic equations*

We define the velocity of the Bose-condensate

$$\boldsymbol{v}_s = \frac{1}{m_B}\nabla\chi. \tag{20}$$

The state of thermodynamic equilibrium ($\eta = \eta^* = U = 0$) is characterized by parameters ϱ^m, θ, c, $v_s^{(\alpha)}$ and $v_n^{(\alpha)}$. The expectation values will be designated only by the velocities, *i.e.* they will be written in the form

$$\langle\quad\rangle_{v_s,v_n}. \tag{21}$$

We can perform the Galilean transformation for Bose and Fermi amplitudes of the form

$$\psi(t, r)\to\psi(t, r)e^{im_B vr}$$

$$\tilde{\psi}(t, x)\to\tilde{\psi}(t, x)e^{im_F vr}. \tag{22}$$

After the transformation we have the expectation values (21) in the form

$$\langle \quad \rangle_{v_s - v, v_n - v}. \tag{23}$$

We are interested especially in the case when $\boldsymbol{v} = \boldsymbol{v}_n$. Hence, the expectation values are considered in the coordinate system with resting normal component ($\langle \quad \rangle_{v_s - v_n, 0}$). If we consider the expectation values from the products of ψ and ψ^+ operators (the number of ψ is equal to the number of ψ^+), we then have $\langle \quad \rangle_{v_s, v_n} = \langle \quad \rangle_{v_s - v_n, 0}$. If we consider a normal system with one velocity ($\langle \quad \rangle_{v_n}$), after the Galilean transformation for $\boldsymbol{v} = \boldsymbol{v}_n$ we go to the system at rest ($\langle \quad \rangle_0$).

Since we consider a normal Fermi system (without the pair condensate and without an additional velocity describing its motion), the expectation values from products of Fermi operators and their derivatives are of the type $\langle \quad \rangle_{v_n}$ and $\langle \quad \rangle_0$.

After using the transformation (22) we have for (10) and (11)

$$\boldsymbol{j}(t, r) = \langle \hat{\boldsymbol{j}} \rangle_{v_s - v, v_n - v} + (m_B \varrho_B + m_F \varrho_F) \boldsymbol{v} = \langle \hat{\boldsymbol{j}} \rangle_{v_s - v, v_n - v} + \varrho^m \boldsymbol{v},$$

$$\varrho^m \varepsilon(t, r) = (\varrho^m \varepsilon)_{v_s - v, v_n - v} + \varrho^m \frac{v^2}{2} \tag{24}$$

where ϱ^m is given by (9). We see that in the case of two kinds of particles with different masses it is necessary to introduce the mass density ϱ^m instead of the density of particles ϱ.

As yet we define only one velocity with help of (20). Hence, let us consider a system with one velocity $\boldsymbol{v}_s$, putting $\boldsymbol{v}_n = 0$. In this case we have, *e.g.*,

$$a(\varrho^m, \theta, c, u) = \text{const.}, \qquad F = F(\varrho, \theta, c, u), \qquad u = \frac{v_s^2}{2} \tag{25}$$

where a is the amplitude of $\langle \psi(t, r) \rangle$ and F is free energy per unit mass.

We define the current

$$j_\alpha = \varrho^m \frac{\partial F}{\partial v_s^{(\alpha)}} = \varrho^m \frac{\partial F}{\partial u} v_s^{(\alpha)} \tag{26}$$

and put

$$\varrho^m \frac{\partial F}{\partial u} = \varrho_s m_B = \varrho_s^m. \tag{27}$$

Formula (27) is the definition of the mass density of the superfluid component ϱ_s^m. We have now

$$j_\alpha = m_B \varrho_s v_s^{(\alpha)} = \varrho_s^m v_s^{(\alpha)}. \tag{28}$$

We wish to demonstrate that the definition (26) is equivalent to the usual definition of the current. For this reason it is convenient to switch to a new coordinate system moving with velocity $-\boldsymbol{v}_s (\boldsymbol{v}_s' = 0,\ \boldsymbol{v}_n' = -\boldsymbol{v}_s)$:

$$\psi(t, r) \to \psi(t, r) e^{-i m_B v_s r}, \qquad \psi(t, x) \to \psi(t, x) e^{-i m_F v_s r}. \tag{29}$$

Since

$$\varrho^m \frac{\partial F}{\partial v_s^{(\alpha)}} = \frac{1}{V} \left\langle \frac{\partial \hat{H}}{\partial v_s^{(\alpha)}} \right\rangle_{0,\,-v_s} \tag{30}$$

(see [4] and [8]), we are interested in the part of the Hamiltonian $\hat{H}(v_s)$ depending on $\boldsymbol{v}_s$:

$$\hat{H}(v_s) = \frac{1}{2\,m_B} \int \sum_{\alpha} \left(\frac{\partial \psi^+}{\partial r_\alpha} - i m_B v_s^{(\alpha)} \psi^+ \right) \left(\frac{\partial \psi}{\partial r_\alpha} + i m_B v_s^{(\alpha)} \psi \right) d\boldsymbol{r} +$$

$$+ \frac{1}{2\,m_F} \int \sum_{s,\alpha} \left(\frac{\partial \tilde{\psi}^+}{\partial r_\alpha} - i m_F v_s^{(\alpha)} \tilde{\psi} \right) \left(\frac{\partial \tilde{\psi}}{\partial r_\alpha} + i m_F v_s^{(\alpha)} \tilde{\psi} \right) d\boldsymbol{r}. \tag{31}$$

Hence,

$$\varrho^m \frac{\partial F}{\partial v_s^{(\alpha)}} = \frac{i}{2V} \left\langle \frac{\partial \psi^+}{\partial r_\alpha} \psi - \psi^+ \frac{\partial \psi}{\partial r_\alpha} \right\rangle_{v_s,0} \int d\boldsymbol{r} +$$

$$+ \frac{i}{2V} \sum_s \left\langle \frac{\partial \tilde{\psi}^+}{\partial r_\alpha} \tilde{\psi} - \tilde{\psi}^+ \frac{\partial \tilde{\psi}}{\partial r_\alpha} \right\rangle_0 \int d\boldsymbol{r} \tag{32}$$

$$= \frac{i}{2} \left\langle \frac{\partial \psi^+(t, r)}{\partial r_\alpha} \psi(t, r) - \psi^+(t, r) \frac{\partial \psi(t, r)}{\partial r_\alpha} \right\rangle_{v_s,0} = \varrho_s^m v_s^{(\alpha)}.$$

The expectation value $\langle \quad \rangle_0$ is isotropic because it does not depend on the vector $\boldsymbol{v}$, therefore, it must be invariant with respect to the transformation $\boldsymbol{r} \to -\boldsymbol{r}$. For this reason the second term in (32) vanishes. The expectation value $\langle \quad \rangle_{v_s,0}$ is not isotropic because the direction of $\boldsymbol{v}_s$ is singlet out. Moreover, the expectation value $\langle \quad \rangle_0$ does not depend on $\boldsymbol{r}$ at equilibrium; therefore, the integration gives a factor V.

For the phase and amplitude of $\langle \psi \rangle = \Phi = a \exp[i\chi]$ we have

$$\frac{\partial \chi}{\partial t} = \lambda^B + \frac{\nabla^2 a}{2\,m_B a} - \frac{m_B v_s^2}{2} + U(t, r) - \frac{\zeta^* + \zeta}{2\,a} -$$

$$- \frac{1}{2\,a^2} \int V(R) \{ X_t^{BF}(r, R) + X_t^{BF*}(r, R) \} d\boldsymbol{R},$$

$$X_t^{BF}(r, r' - r) = \langle \hat{\varrho}(t, r') \rangle \langle \psi^+ \rangle,$$

$$\zeta(t, r) = \eta(t, r) e^{-i\chi(t,r)} \tag{33}$$

and

$$\frac{\partial \varrho_c}{\partial t} + \operatorname{div}(\varrho_c \boldsymbol{v}_s) = i \int V(R) [X_t^{BF}(r, R) - X_t^{BF*}(r, R)] d\boldsymbol{R} + i \sqrt{\varrho_c} (\zeta^* - \zeta), \quad \varrho_c = a^2. \tag{34}$$

For thermodynamic equilibrium ($U = 0$, $\eta = 0$) we have

$$\frac{1}{2a^2}\int V(R)[X^{BF}(R|\varrho^m, \theta, c, v_s) + X^{BF*}(R|\varrho^m, \theta, c, v_s)]d\boldsymbol{R}$$

$$= m_B\mu_B(\varrho^m, \theta, c, u) - \frac{m_B v_s^2}{2} = \frac{1}{a^2}\int V(R)X^{BF}d\boldsymbol{R} = \frac{1}{a^2}\int V(R)X^{BF*}d\boldsymbol{R} \tag{35}$$

where μ_B is chemical potential per unit mass for Bose particles.

The free energy of our system is equal to $MF(\varrho^m, \theta, c, u)$, where

$$M = V(m_B\varrho_B + m_F\varrho_F) = V(\varrho_B^m + \varrho_F^m) = V\varrho^m \tag{36}$$

and F is free energy per unit mass.

Now we derive some equations for the chemical potential. We have

$$\frac{\partial(MF)}{(V\varrho_B^m)} = \left(\frac{\partial(\varrho^m F)}{\partial\varrho_B^m}\right)_{\theta,c} = \left(\frac{\partial(\varrho^m F)}{\partial\varrho^m}\right)_{\theta,c}\frac{\partial\varrho^m}{\partial\varrho_B^m} + \left(\frac{\partial(\varrho^m F)}{\partial c}\right)_{\theta,\varrho^m}\frac{\partial c}{\partial\varrho_B^m}$$

$$= \mu_B = \mu - \frac{c}{\varrho^m}\left(\frac{\partial(\varrho^m F)}{\partial c}\right)_{\theta,\varrho^m}, \tag{37}$$

$$\frac{\partial(MF)}{\partial(V\varrho_F^m)} = \left(\frac{\partial(\varrho^m F)}{\partial\varrho_F^m}\right)_{\theta,c} = \left(\frac{\partial(\varrho^m F)}{\partial\varrho^m}\right)_{\theta,c}\frac{\partial\varrho^m}{\partial\varrho_F^m} + \left(\frac{\partial(\varrho^m F)}{\partial c}\right)_{\theta,\varrho^m}\frac{\partial c}{\partial\varrho_F^m}$$

$$= \mu_F = \mu + \frac{1-c}{\varrho^m}\left(\frac{\partial(\varrho^m F)}{\partial c}\right)_{\theta,\varrho^m}. \tag{38}$$

From (37) and (38) we have

$$\mu = c\mu_F + (1-c)\mu_B = \left(\frac{\partial(\varrho^m F)}{\partial\varrho^m}\right)_{\theta,c}, \tag{39}$$

and

$$\mu_F - \mu_B = \left(\frac{\partial F}{\partial c}\right)_{\varrho^m,\theta} \equiv z, \qquad c = \frac{\varrho_F^m}{\varrho_B^m + \varrho_F^m} \tag{40}$$

$$\mu_B = \mu - zc$$

(in Khalatnikov's notation $z = Z/\varrho^m$, see [5]).

We are interested in the hydrodynamic nonequilibrium processes. In this case the expectation values can be expanded into a series with respect to the space derivatives of local variables $\varrho^m(t, r)$, $\theta(t, r)$, $c(t, r)$, $v^{(\alpha)}(t, r)$ and $v_n^{(\alpha)}(t, r)$. We consider the derivatives $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \boldsymbol{r}}$ as quantities of the first order of smallness, the derivatives like $\frac{\partial^2}{\partial r_\alpha \partial r_\beta}$ as quantities of the second order of smallness, *etc.* The first term in the series does not depend on derivatives (see, *e.g.*, [4] and [8]). Using this procedure, which enables us to keep only the terms of the same order of smallness, we can write in (18)

$$\frac{1}{2}\int \frac{\partial V(R)}{\partial R_\alpha}\,[\tilde{\mathscr{D}}_t(r,\,-R)+\tilde{\mathscr{D}}_t(r-R,\,R)]d\boldsymbol{R}$$

$$=\frac{1}{2}\int \frac{\partial V(R)}{\partial R_\alpha}\left[\tilde{\mathscr{D}}_t(r,\,-R)+\tilde{\mathscr{D}}_t(r,\,R)-\sum_\beta \frac{\partial \tilde{\mathscr{D}}_t}{\partial r_\beta}R_\beta++\right]d\boldsymbol{R} \tag{41}$$

$$=\sum_\beta \frac{\partial}{\partial r_\beta}\left[-\frac{1}{2}\int \frac{\partial V(R)}{\partial R_\alpha}R_\beta\tilde{\mathscr{D}}_t d\boldsymbol{R}\right].$$

Now (18) can be written as

$$\frac{\partial j_\alpha(t,r)}{\partial t}=\sum_\beta \frac{\partial T_{\alpha\beta}}{\partial r_\beta}+\varrho_B\frac{\partial}{\partial r_\alpha}(\lambda^B-U)+\varrho_F\frac{\partial}{\partial r_\alpha}(\lambda^F-U)+$$

$$+\frac{1}{2}\left(\frac{\partial \Phi^*}{\partial r_\alpha}\eta+\frac{\partial \Phi}{\partial r_\alpha}\eta^*-\Phi^*\frac{\partial \eta}{\partial r_\alpha}-\Phi\frac{\partial \eta^*}{\partial r_\alpha}\right) \tag{42}$$

where

$$T_{\alpha\beta}(\varrho^m,\,\theta,\,c,\,v_s)=-\frac{1}{2\,m_B}\left\langle\frac{\partial \psi^+}{\partial r_\alpha}\frac{\partial \psi}{\partial r_\beta}+\frac{\partial \psi^+}{\partial r_\beta}\frac{\partial \psi}{\partial r_\alpha}\right\rangle_{v_s,0}-$$

$$-\frac{1}{2\,m_F}\sum_s\left\langle\frac{\partial \tilde{\psi}^+}{\partial r_\alpha}\frac{\partial \tilde{\psi}}{\partial r_\beta}+\frac{\partial \tilde{\psi}^+}{\partial r_\beta}\frac{\partial \tilde{\psi}}{\partial r_\alpha}\right\rangle_0+$$

$$+\frac{1}{2}\int \frac{\partial V(R)}{\partial R_\alpha}R_\beta\tilde{\mathscr{D}}(R|\varrho^m,\,\theta,\,c,\,v_s)d\boldsymbol{R}=\overset{m}{\varrho}_B v_s^{(\alpha)}v_s^{(\beta)}-\delta_{\alpha\beta}\mathscr{P}. \tag{43}$$

$\tilde{\mathscr{D}}(R|\varrho^m,\,\theta,\,c,\,v_s)$ is the "zero" term in the expansion of $\tilde{\mathscr{D}}_t(r,\,R)$ into a series. Formula (43) was obtained with the help of the identity

$$\mathscr{P}=-\frac{\partial\left[MF\left(\frac{M}{V},\,\theta,\,c\right)\right]}{\partial V}=\frac{M}{V^2}M\frac{\partial F}{\partial \varrho^m}=(\varrho^m)^2\frac{\partial F}{\partial \varrho^m}. \tag{44}$$

Now let us consider a system with two independent velocities $\boldsymbol{v}_s$ and $\boldsymbol{v}_n$. We notice that the functions $\mathscr{D}$ (18) and X (33) remain unchanged after the Galilean transformation. We perform now the Galilean transformation (22) for $\boldsymbol{v}=\boldsymbol{v}_n$.

The general expression for the current has the form (see (32))

$$j_\alpha=\frac{i}{2}\left\langle\frac{\partial \psi^+}{\partial r_\alpha}\psi-\psi^+\frac{\partial \psi}{\partial r_\alpha}\right\rangle_{v_s,v_n}+\frac{i}{2}\sum_s\left\langle\frac{\partial \tilde{\psi}^+}{\partial r_\alpha}\tilde{\psi}-\tilde{\psi}^+\frac{\partial \tilde{\psi}}{\partial r_\alpha}\right\rangle_{v_n}$$

$$=\frac{i}{2}\left\langle\left(\frac{\partial \psi^+}{\partial r_\alpha}-im_B v_n^{(\alpha)}\psi^+\right)\psi-\psi^+\left(\frac{\partial \psi}{\partial r_\alpha}+im_B v_n^{(\alpha)}\psi\right)\right\rangle_{v_s-v_n,0}+$$

$$+\frac{i}{2}\sum_s\left\langle\left(\frac{\partial \tilde{\psi}^+}{\partial r_\alpha}-im_F v_n^{(\alpha)}\tilde{\psi}^+\right)\tilde{\psi}-\tilde{\psi}^+\left(\frac{\partial \tilde{\psi}}{\partial r_\alpha}+im_F v_n^{(\alpha)}\tilde{\psi}\right)\right\rangle_0$$

$$=\overset{m}{\varrho}_B v_n^{(\alpha)}+\overset{m}{\varrho}_F v_n^{(\alpha)}+\overset{m}{\varrho}_s(v_s^{(\alpha)}-v_n^{(\alpha)}). \tag{45}$$

We define now the mass density of the normal component by

$$\varrho_n^m = \varrho^m - \varrho_s^m \tag{46}$$

and get

$$j_\alpha = \varrho_n^m v_n^{(\alpha)} + \varrho_s^m v_s^{(\alpha)}. \tag{47}$$

This expression was obtained for a constant velocity v_n. Now we treat equation (47) as a definition of the local velocity v_n.

For the stress tensor in the case of two velocities we have the expression

$$T_{\alpha\beta}(\varrho^m, \theta, c, v_s, v_n) = -\delta_{\alpha\beta}\mathscr{P}(\varrho^m, \theta, c) - \\ -\varrho_s^m v_s^{(\alpha)} v_s^{(\beta)} - \varrho_n^m v_n^{(\alpha)} v_n^{(\beta)}. \tag{48}$$

Now we are interested in equation (19). This equation contains a term of the form considered in (41) (only instead of $\mathscr{D}$ we have now function G). After the same procedure as used in (41) we can write (19) in the form

$$\frac{\partial \varrho^m \varepsilon}{\partial t} = \sum_\alpha \frac{\partial I_\alpha}{\partial r_\alpha} - \frac{1}{m_B}\sum_\alpha j_B^{(\alpha)} \frac{\partial}{\partial r_\alpha}(U - \lambda^B) - \frac{1}{m_F}\sum_\alpha j_F^{(\alpha)} \frac{\partial}{\partial r_\alpha}(U - \lambda^F) + \\ + \frac{i}{4 m_B}(\eta \nabla^2 \Phi^* - \eta^* \nabla^2 \Phi + \Phi^* \nabla^2 \eta - \Phi \nabla^2 \eta^*) + \\ + \frac{i}{2}\int V(r - r')\,[\eta^*(t, r)\langle \hat{\varrho}(t, r')\psi(t, r)\rangle + \eta^*(t, r')\langle \hat{\varrho}(t, r)\psi(t, r')\rangle - \\ - \eta(t, r')\langle \psi^+(t, r')\hat{\varrho}(t, r)\rangle - \eta(t, r)\langle \psi^+(t, r)\hat{\varrho}(t, r')\rangle]\,d\boldsymbol{r}' \tag{49}$$

where for the energy current I_α we have

$$I_\alpha(\varrho^m, \theta, c, v_s, v_n) = I_\alpha(\varrho^m, \theta, c, v_s - v_n, 0) - \\ - v_n^{(\alpha)}\left[\varrho^m E(\varrho^m, \theta, c, u) + \frac{\varrho^m v_n^2}{2} + \varrho_s^m(\boldsymbol{v}_s - \boldsymbol{v}_n)\boldsymbol{v}_n\right] + \\ + \sum_\beta v_n^{(\beta)} T_{\alpha\beta}(\varrho^m, \theta, c, v_s - v_n, 0) - \frac{v_n^2}{2}\varrho_s^m(v_s^{(\alpha)} - v_n^{(\alpha)}),$$

$$I_\alpha(\varrho^m, \theta, c, v_s - v_n, 0) = -\frac{i}{4 m_B}\left\langle (\nabla^2 \psi^+)\frac{\partial \psi}{\partial r_\alpha} - \frac{\partial \psi^+}{\partial r_\alpha}(\nabla^2 \psi)\right\rangle_{v_s - v_n, 0} + \\ + \sum_\beta \int \left(V(R)\delta_{\alpha\beta} - \frac{\partial V}{\partial R_\beta} R_\alpha\right)\tilde{G}^{(\beta)}(R|\varrho^m, \theta, c, v_s - v_n, 0)\,d\boldsymbol{R},$$

$$T_{\alpha\beta}(\varrho_m, \theta, c, v_s - v_n, 0) = -\mathscr{P}\delta_{\alpha\beta} - \varrho_s^m(v_s^{(\alpha)} - v_n^{(\alpha)})(v_s^{(\beta)} - v_n^{(\beta)}). \tag{50}$$

In these calculations the same procedure as in (41) was applied. In order to rewrite the last terms of (49) see (35).

We assume that (see [4])

$$I_\alpha(\varrho^m, \theta, c, v_s - v_n, 0) = -\mu_B \varrho_s^m (v_s^{(\alpha)} - v_n^{(\alpha)}), \tag{51}$$

use the equation

$$\varrho^m \varepsilon = \varrho^m E(\varrho^m, \theta, c, u) + \frac{\varrho^m v_n^2}{2} + \varrho_s^m (\boldsymbol{v}_s - \boldsymbol{v}_n)\boldsymbol{v}_n \tag{52}$$

and have finally

$$\frac{\partial}{\partial t}\left[\varrho^m E + \frac{\varrho^m v_n^2}{2} + \varrho_s^m(\boldsymbol{v}_s - \boldsymbol{v}_n)\boldsymbol{v}_n\right] + \sum_\beta \frac{\partial}{\partial r_\beta}\left\{v_n^{(\beta)}\left[\frac{\varrho^m v_n^2}{2} + \varrho^m E + \mathscr{P} + \varrho_s^m(\boldsymbol{v}_s - \boldsymbol{v}_n)\boldsymbol{v}_n\right] + \right.$$
$$\left. + (v_s^{(\beta)} - v_n^{(\beta)})\varrho_s^m\left[\mu_B + \boldsymbol{v}_n\left(\boldsymbol{v}_s - \frac{\boldsymbol{v}_n}{2}\right)\right]\right\} = -\sum_\beta j_B^{(\beta)} \frac{1}{m_B}\frac{\partial U}{\partial r_\beta} -$$
$$- \sum_\beta j_F^{(\beta)} \frac{1}{m_F}\frac{\partial U}{\partial r_\beta} + i m_B(\zeta^* - \zeta)\left[\frac{v_s^2}{2} + \mu_B + \frac{(\boldsymbol{v}_s - \boldsymbol{v}_n)^2}{2}\right]. \tag{53}$$

Now we introduce entropy per unit mass

$$S = -\frac{\partial F}{\partial \theta}. \tag{54}$$

Thanks to this formula we can introduce the equation for entropy conservation instead of the equation for energy conservation (53) (see [4] and [8]).

The full system of hydrodynamic equations for dilute solutions with external potential and "sources" has the form

$$\frac{\partial \varrho^m}{\partial t} + \operatorname{div} \boldsymbol{j} = i m_B a(\zeta^* - \zeta), \tag{55}$$

$$\frac{\partial (c\varrho^m)}{\partial t} + \operatorname{div}(c\varrho^m \boldsymbol{v}_n) = 0, \tag{56}$$

$$\frac{\partial}{\partial t}(\varrho_s^m v_s^{(\alpha)} + \varrho_n^m v_n^{(\alpha)}) + \frac{\partial \mathscr{P}}{\partial r_\alpha} + \sum_\beta \frac{\partial}{\partial r_\beta}(\varrho_n^m v_n^{(\alpha)} v_n^{(\beta)} + \varrho_s^m v_s^{(\alpha)} v_s^{(\beta)}) = i m_B a v_s^{(\alpha)}(\zeta^* - \zeta) -$$
$$- \left(\varrho_B^m \frac{1}{m_B} + \varrho_F^m \frac{1}{m_F}\right)\frac{\partial U}{\partial r_\alpha}, \tag{57}$$

$$\frac{\partial v_s^{(\alpha)}}{\partial t} = -\frac{\partial}{\partial r_\alpha}\left[\frac{v_s^2}{2} - \frac{(\boldsymbol{v}_s - \boldsymbol{v}_n)^2}{2} + \mu_B + \frac{1}{m_B} U\right], \tag{58}$$

$$\frac{\partial (\varrho^m S)}{\partial t} + \operatorname{div}(S\varrho^m \boldsymbol{v}_n) = 0. \tag{59}$$

Equations (55)—(59), without external potential U and "sources" ζ and ζ^*, were first derived by Khalatnikov [5] starting out from phenomenological consideration.

REFERENCES

[1] P. C. Hohenberg, P. C. Martin, *Ann. Phys. (N.Y.)*, **34**, 291 1965.
[2] L. D. Landau, *J. Phys.*, **5**, 71 (1941).
[3] L. D. Landau, *J. Phys.*, **11**, 91 (1947).
[4] N. N. Bogolyubov, *On the Hydrodynamics of a Superfluid* (in Russian), Dubna preprint 1963.
[5] I. M. Khalatnikov, *JETP.*, **23**, 169 (1952); *Vvedenie v teoriu sverkhtekutsesti*, Izd. Nauka, Moskva 1965; English transl. *Introduction to the Theory of Superfluidity*, Benjamin, New York 1966.
[6] Z. M. Galasiewicz, *Acta Univ. Wratislaviensis*, No **141** (1970), Materials of the VII-th Winter School of Theoretical Physics in Karpacz "Liquid Helium and Many Body Problems".
[7] J. Bardeen, G. Baym, D. Pines, *Phys. Rev.*, **156**, 207 (1967).
[8] Z. M. Galasiewicz, *Superconductivity and Quantum Fluids*, Pergamon Press, Oxford 1970.