A GENERAL "DYNAMICAL" MODEL FOR A CLASS OF STATISTICAL DISTRIBUTIONS (I)

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A simple differential equation is deduced in order to show that a "dynamical" hypothesis can be assumed at the basis of the so-called *generalized gamma distribution* (Lienhard, J. H., Meyer, P. L., *Quart Appl. Math.*, **25**, 330 (1967)).

Moreover, it is verified that, under a natural specialization, its general solution contains, as particular cases, results from Lienhard and Meyer paper hitherto pursued by classical Boltzmann's methods.

Finally, an undeniable formal analogy of foundations with Schrödinger approach of Quantum Mechanics is underlined.

(Criticism and interpretation of such an analogy are delayed to a further note).

The so-called *generalized gamma distribution* [1] is a probability density function containing, as particular cases, a wide class of well-known statistical distributions, all depending on a continuous variable t.

It is assumed to represent a physical-statistical situation of a very general feature, *i.e.* what physicist calls "*the model*", and it applies to a large collection of peculiar problems, such as those leading to the *Weibull distribution*, to the *hydrograph distribution* [2], to the *gamma distribution*, to the *Rayleigh distribution*, to the "*first*" and "*second*" *Maxwell distributions* [3], to the *exponential distribution*.

Lienhard and Meyer have deduced [1], under assumption of some reasonable requirements upon the model, a very interesting form depending on three parameters and giving, for various choices of two among them, all the mentioned distributions.

These authors follow, as it is customary, a classical Boltzmann's procedure, whose general lines are reported within the first paragraph of the present note; there the reader will note the generality of the assumptions made by Lienhard and Meyer.

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Nevertheless, it appears desirable to construct, as a basis for a generalized theory, some "*dynamical*" hypothesis, that is *to define a model by means of some law to which it succumbs and connecting causes and effects.*

This is the subject of the present paper, although it is not the only aim.

In effect, the first stage is that of carrying out the law in terms of a very simple differential equation, whose general solution contains, as a special case, the *generalized gamma distribution* (what we shall abbreviate by g.g.d.).

In addition a remarkable analogy of foundation with the differential operator structure of Quantum Mechanics will be pointed out, which in this work is suggested as a possible starting point for a further coalescence of Statistical Mechanics.

(We shall use the symbol $P = P(t)$ to represent the probability regarded as a function of the unspecified variable t, while

$$\psi(t) = \frac{dP}{dt}$$

will be the probability density function. For symbols not defined we refer to [1]).

1. We report, as announced earlier, a synthesis of the procedure followed by the mentioned authors [1].

Let S be a physical system obeying the law of large numbers and let x be a given event which can occur in S.

Besides, let t be a continuous variable of general meaning, on which the probability P depends; (it could represent the time during which the system was subjected to a constant stimulation, as well as the amount of the stimulation itself, or any other quantity directly connected with the value of the probability. For the sake of brevity, however, one speaks of t as of *time*. This does not cause any loss of generality).

Suppose, now, that the event x occurs N_i times within the interval $[t_{i-1}, t_i)$ and set:

$$\Delta t = t_i - t_{i-1}$$

Finally, assume what follows:

a) the total number of events is fixed:

$$\sum_{i=1}^{\infty} N_i = N;$$

b) for any β there is a positive constant K such that:

$$\sum_{i=1}^{\infty} \frac{N_i}{N} t_i^{\beta} = K;$$

c) the event x can occur, within the interval $[t_{i-1}, t_i)$, in a number g_i of distinct ways that results proportional to a fixed power of t_i:

$$g_i = A t_i^{\alpha-1}$$

(Constants α and β are assumed positive).

Under these assumptions, the g.g.d. is carried out by means of the well-known technique of maximizing the probability of the distribution.

After calculations and denoting by $\bar{N}_i$ these values of N_i which maximize the above said probability, one has:

$$\frac{\bar{N}_i}{N} = \Delta t \frac{\beta(\beta K/\alpha)^{-\alpha/\beta}}{\Gamma(\alpha/\beta)} t_i^{\alpha-1} \exp\left(-\alpha/\beta \frac{t_i^\beta}{K}\right)$$

which does not depend on the constant A introduced before.

Afterwards, through a passage to a continuous variable, the probability density function (t) of detecting the given event x in S is deduced and it results:

$$f(t) = H_1 t^{\alpha-1} \exp\left[-(t/H_2)^\beta\right],$$

where H_1 and H_2 depend on α, β and K only.

(It is worth noting that the parameter K is completely unessential, since it cancels after a suitable normalization).

Concluding, for various choices of α and β, one obtains from $f(t)$ the following distributions:

1) *the Weibull distribution* ($\alpha = \beta$);
2) *the hydrograph distribution* [2] ($\beta = 2$);
3) *the gamma distribution* ($\beta = 1$);
4) *the Rayleigh distribution* ($\alpha = \beta = 2$);
5) *the "first" Maxwell distribution* ($\alpha = 3, \beta = 2$);
6) *the "second" Maxwell distribution* ($\alpha = 1, \beta = 2$);
7) *the exponential distribution* ($\alpha = \beta = 1$).

2. The "dynamical" model. We now intend to suggest a differential hypothesis about the law that one can assume to represent the mechanism of the probabilistic evolution (or behaviour) of the system S under a specified stimulation.

As the reader can see, such a law will result by means of very simple considerations, all arising from quite general statements about the assumed model.

Then, let t_0 be a fixed value of time t and let

$$(dP)_0 = \psi(t_0)dt$$

be the probability of detecting the event x in S at the instant t_0.

At the instant $t_0+\Delta t$ this probability becomes:

$$(dP)_{0,\,\Delta t} = \psi(t_0+\Delta t)dt;$$

so that we can define a *relative variation of probability* as follows:

$$v(t_0, \Delta t) = \frac{\psi(t_0+\Delta t)-\psi(t_0)}{\psi(t_0)}. \qquad (1)$$

Let us now denominate by $H(t_0)$ the sum of all stimulations to which S is subjected at time t_0; (in a general situation H consists of two kinds of coarse terms: one which has the effect of enhancing that variation, the other which tends to lessen it).

At present we are able to formulate the following general law:

the relative variation of probability is proportional, within an infinitesimal $\varepsilon(\Delta t)$ *of higher order than* Δt*, to the sum of all stimulations and to the interval of time* Δt.

In quantitative terms:

$$v(t_0, \Delta t) = kH(t_0)\Delta t + \varepsilon(\Delta t). \tag{2}$$

Hence, letting $\Delta t \to 0$ and after a slight manipulation, we have at any time t:

$$\frac{d\psi}{dt} = kH\psi, \tag{3}$$

where coefficient k depends, eventually, on the system S only.

We shall call equation (3) the equation of *the most general gamma distribution.* (Abbreviated m.g.g.d.).

In effect, as we are going to show, (3) gives place to the g.g.d. under a reasonable specialization of $H(t)$. But, in addition, one can draw from (3) a wider class of distributions, all corresponding to the general model we have assumed.

It will be matter of further investigation to exhaust the entire field of application of the hypothesis expressed by (3).

Furthermore, we shall then be concerned with the completion of theory, since the case of non-fixed event, (introduced here in paragraph 4.), will be interpreted in detail with regard to equation (7) of this note, which is essentially the reduction to Schrödinger formalism.

3. We want now to specialize equation (3) in order to show that it contains the results of Lienhard and Meyer [1] about the g.g.d..

Therefore, suppose that $kH(t)$ consists in the following two terms:

$$\begin{cases} k_1 t^a, & a \geqslant 0, \quad \text{(the "enhancing term")} \\ k_2 t^{-1}, & \text{(the "lessening term")}, \end{cases} \tag{4}$$

k_1 and k_2 being constant coefficients which are assumed, *a priori*, arbitrarily at all.

Substitution of (4) into (3) and consequent integration lead to a probability density function expressed by:

$$\psi(t) = ct^{k_2} \exp(k' t^{b+1}) \tag{5}$$

where c is the arbitrary integration constant and where we have put:

$$k' = \frac{k_1}{a+1}.$$

(Furthermore, note that c will be no more arbitrary after normalization).

And since normalization requirements lead to the assumption of negative values of k', we settle $h_1 > 0$ such that:

$$k' = -h_1,$$

while, for homogeneousness of notation, we shall write h_2 instead of k_2.

Hence equation (5) reads:

$$\psi(t) = ct^{h_2} \exp(-h_1 t^{a+1}). \tag{5 bis}$$

This furnishes essentially a class of two-parameter distributions, since for various choices of h_2 and a (h_1 being automatically involved in the normalization procedure), it gives place to the following situations:

1) the Weibull distribution for: $h_2 = a = \alpha - 1$;
2) the hydrograph distribution for: $h_2 = \alpha - 1, \quad a = 1$;
3) the gamma distribution for: $h_2 = \alpha - 1, \quad a = 0$;
4) the Rayleigh distribution for: $h_2 = a = 1$;
5) the "first" Maxwell distribution for: $h_2 = 2, \quad a = 1$;
6) the "second" Maxwell distribution for: $h_2 = 0, \quad a = 1$;
7) the exponential distribution for: $h_2 = a = 0$.

(It would be worthwhile to recall that distributions from 1) to 7) are not all independent from one another).

4. Now, suppose that the expression:

"*it occurs in S the event x*"

means:

"*the physical quantity* (that is the observable) *X takes in S the value x*".

Suppose, too, that the value of such an observable (generally regarded as a vector) can vary continuously over a given domain D (generally assumed to be n-dimensional).

In order to give a mathematical sense to this assumption, it is sufficient to understand $D \subset B$, B being a Banach space.

In such a connection, one must generalize equation (3) by the imposition that ψ, as well as k and H, may depend on x; and one must write, in general:

$$\frac{\partial \psi(x, t)}{\partial t} = k(x) H(x, t) \psi(x, t). \tag{6}$$

Furthermore, to suppose the action of stimulations simply expressed by a multiplicative factor (consisting in a function) will result in a too strong restriction, which is not acceptable in a general context.

It seems natural to regard $H(x, t)$ as an operator acting in the space of the functions $\psi(x, t)$, while it is possible to preserve the nature of a constant to $k(x)$, inserting its eventual dependence on x into $H(x, t)$.

Summarizing the above considerations, the reader could easily recognize a very familiar differential equation at the basis of our generalized statistical model, *i.e.*:

$$ih \frac{\partial \psi}{\partial t} = H\psi \tag{7}$$

where, in order to enhance the formal analogy with Schrödinger equation, the following position has been made: $h = -i/k$.

Obviously h is, so far, completely general.

One may observe that ψ involved in the very Schrödinger equation is not a probability density function, for there one assumes:

$$|\psi|^2 = \frac{dP}{dt}^{1}, \tag{*}$$

the remaining variables being fixed.

However, it has to be pointed that if we repeat arguments of paragraph *2.* substituting ψ by $|\psi|^2$ (that is by ψ^2, since ψ is a real function), we obtain:

$$\frac{d\psi^2}{dt} = kH\psi^2, \quad \text{with} \quad \psi^2 = \frac{dP}{dt} \tag{8}$$

which means:

$$2\psi \frac{d\psi}{dt} = kH\psi^2 \tag{9}$$

and (9) reduces to (3) apart from the unessential factor 2 of the first member.

Therefore the reader could follow this entire paper under the replacement of (*) for ψ.

Nevertheless, although it is matter of the following paper, we have to notice that difficulties arise when we give to H an operator meaning.

In fact at this step (3) and (9) will be no more equivalent: to extend (9) *before* division by ψ leads to a *non-linear* differential equation for ψ itself; to do it *after* division leads, on the contrary, to a *linear* one.

Thus the question arises: *have we to assume a superposition principle for the probability density function or for its square root?*

Or, which is the same thing: *have we to proceed along the line of quantum-mechanical formalism* (second choice) *or along an independent one* (first choice)?

Moreover, in the second case one may no more restrict oneself to real functions and a further stage ought to be performed to pass to the complex field.

REFERENCES

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[3] S. Chapman and T. G. Cowling, *The mathematical theory of non-uniform gases*, Cambridge University Press, London 1960, Section 4.

[1] Remember that t *does not necessarily* represent the variable "time". Thus, it may be a position variable on a line.