Vol. A38 (1970) ACTA PHYSICA POLONICA Fasc. 5

APPROACH TO THE QUANTUM MECHANICAL FOUR-BODY BOUND--STATE PROBLEM. I. DISTINGUISHABLE PARTICKLES

By E. Magyari

Department of Theoretical Physics, University of Craiova*

(*Received March 14, 1970*)

The aim of the present paper is to sketch how an earlier approach to the three-body bound--state problem, developed by Eyges and Jasperse and Friedman, may be generalized to the N-body case, and to develop it in detail for a system of four distinguishable particles, having in view that this case, in contrast to the three-body one, obviously reflects the algorithm of the proposed generalization.

1. *Introduction*

In the last years much attention has been paid to the few-particle quantum systems, due to the broad range of physical effects introduced by the presence of more than two particles.

For the case of a bound-state system of three identical particles, Eyges has shown the opportunity of writing the wave function as a sum of three parts, called "two-body orbitals" or "partialwavefunctions", [1]. Extending this approach, Jasperse and Friedman have shown in their comprehensive paper [2] how solutions for a bound-state system of three arbitrary particles may be constructed which are eigenstates of a complete set of commuting operators. The method has been applied with success by these authors to the helium-like atom and has led to a new technique for calculating bound-state energies for the three-body systems [2].

The essential starting point of the papers [1] and [2] consists in separating off the center--of-mass motion of the three-body system by using the so called Jacobi coordinates:

$$\boldsymbol{R} = (m_i\boldsymbol{r}_i+m_j\boldsymbol{r}_j+m_k\boldsymbol{r}_k)(m_i+m_j+m_k)^{-1}$$

$$\boldsymbol{r}_{ij} = \boldsymbol{r}_i-\boldsymbol{r}_j, \quad \boldsymbol{\rho}_k = \boldsymbol{r}_k-(m_i\boldsymbol{r}_j+m_j\boldsymbol{r}_j)(m_i+m_j)^{-1} \tag{1}$$

and in treating on an equal footing all the three sets of these (linearly dependent !) coordinates, generated by $(i, j, k) = \text{cycl}\,(1, 2, 3)$, [2] or by the restriction $i < j$, [1]. In this way one associates to each pair-potential $V_{ij}(\boldsymbol{r}_{ij})$ a particular Jacobi frame and a "two-body orbital" $\psi_{ij}(\boldsymbol{r}_{ij}, \boldsymbol{\rho}_k)$. The main advantage of this method is that the "orbital" functions have a rapidly convergent partial wave expansion and that the formalism is highly symmetrical.

---

* Address: University of Craiova, Craiova, Romania.

The purpose of the present paper is to sketch, how the method initiated by Eyges may be generalized to the N-body case, and to develop it in detail for the bound-state system of four distinguishable particles governed by the Hamiltonian

$$H=\sum_{i=1}^{4}(p_i^2/2m_i)+\sum_{i<j}V_{ij}(|\boldsymbol{r}_i-\boldsymbol{r}_j|). \qquad (2)$$

The generalization of the Jacobi coordinates may be performed without any difficulty. They arise from a well-known CM-motion separation procedure, applicable to any several particle system whenever the interaction is invariant under an overall translation of all particles [3]. In spite of the simplicity of this procedure, the extension of the formalism to the N-body case ($N>3$) is not trivial. The reason is the following. The above quoted step by step reduction to the center-of-mass of the assembly of distinguishable particles can be performed in $N!$ possible ways, and this leads to $N!$ (linearly dependent!) sets of Jacobi coordinates, a number which exceeds by far that of the $C_N^2=N(N-1)/2$ pair-potentials. But, while in the three-body case we can remove the redundant Jacobi sets by the restriction $i<j$, or by that rather symmetrical one $(i,j,k)=\text{cycl}\,(1,2,3)$, in the N-body case ($N>3$) these rules are not sufficient to pick out the necessary number of C_N^2 sets. Therefore, there is a need to invent a rule of selecting indices, generally applicable and at the same time easy to handle. On the other hand, the foregoing of the cyclical permutation rule led to the loss of a familiar symmetry. This circumstance adds a further inconvenience to the troubles caused by the higher dimensionality of the problem.

The organization of the paper is as follows: In Sec. 2 we outline the kinematics for the N-body case and establish the Schrödinger equation and the associated Green's functions for the four-body problem, using an appropriate number of Jacobi frames. In Sec. 3 we perform the "two-body orbital" decomposition of the wave function, and deduce the set of coupled integral equations for the four-body bound-state problem. The final section is devoted to some considerations about the total angular momentum operator for four and N distinguishable particles in the framework of the outlined formalism. The generalization of the formalism to include identical particles, in order to make it applicable to the bound-state systems such as the hydrogen molecule, α-particles, litium-like atoms, etc., is the subject of a forthcoming paper.

2. *Kinematics, Schrödinger equation and associated Green's functions*

By using the reduction procedure quoted in the preceeding section, we obtain for the N-particle system the following Jacobi coordinates

$$\begin{gathered}\boldsymbol{R}=\Big(\sum_{n=1}^{N}m_{i_n}\boldsymbol{r}_{i_n}\Big)\Big(\sum_{n=1}^{N}m_{i_n}\Big)^{-1}\\ \boldsymbol{r}_{i_1i_2}=\boldsymbol{r}_{i_1}-\boldsymbol{r}_{i_2}\\ \boldsymbol{q}_{i_1i_2\ldots i_k}=\boldsymbol{r}_{i_k}-\Big(\sum_{n=1}^{k-1}m_{i_n}\boldsymbol{r}_{i_n}\Big)\Big(\sum_{n=1}^{k-1}m_{i_n}\Big)^{-1}\\ k=3,4,\ldots,N\end{gathered} \qquad (3)$$

where $i_1, i_2, \ldots, i_N$ are the numbers 1, 2, ..., N in an arbitrary order. For $k = N$, the first $N-1$ indices of $\boldsymbol{q}$ are irrelevant and thus we denote the last coordinate by $\boldsymbol{\rho}_{i_N}$. The relations (3) provide $N!$ generalized Jacobi frames. But, as it is shown in the Appendix, the simultaneous restrictions

$$i_1 < i_2, \; i_3 < i_4, \; i_4 < i_5, \ldots, i_{N-1} < i_N \tag{4}$$

select out the necessary number of $N(N-1)/2$ Jacobi sets.

For our four-body system of distinguishable particles we use the notation

$$\begin{gathered}
\boldsymbol{R} = (m_i\boldsymbol{r}_i + m_j\boldsymbol{r}_j + m_k\boldsymbol{r}_k + m_l\boldsymbol{r}_l)(m_i + m_j + m_k + m_l)^{-1} \\
\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j \\
\boldsymbol{q}_{ijk} = \boldsymbol{r}_k - (m_i\boldsymbol{r}_i + m_j\boldsymbol{r}_j)(m_i + m_j)^{-1} \\
\boldsymbol{\rho}_l = \boldsymbol{r}_l - (m_i\boldsymbol{r}_i + m_j\boldsymbol{r}_j + m_k\boldsymbol{r}_k)(m_i + m_j + m_k)^{-1},
\end{gathered} \tag{5}$$

where i, j, k, l, are 1, 2, 3, 4 in such a way that $i < j$ and $k < l$ simultaneously. Hence, the six Jacobi frames are defined by the following coordinates

$$\begin{array}{ll}
1. \; (\boldsymbol{R}, \boldsymbol{r}_{12}, \boldsymbol{q}_{123}, \boldsymbol{\rho}_4); & 4. \; (\boldsymbol{R}, \boldsymbol{r}_{23}, \boldsymbol{q}_{231}, \boldsymbol{\rho}_4); \\
2. \; (\boldsymbol{R}, \boldsymbol{r}_{13}, \boldsymbol{q}_{132}, \boldsymbol{\rho}_4); & 5. \; (\boldsymbol{R}, \boldsymbol{r}_{24}, \boldsymbol{q}_{241}, \boldsymbol{\rho}_3); \\
3. \; (\boldsymbol{R}, \boldsymbol{r}_{14}, \boldsymbol{q}_{142}, \boldsymbol{\rho}_3); & 6. \; (\boldsymbol{R}, \boldsymbol{r}_{34}, \boldsymbol{q}_{341}, \boldsymbol{\rho}_2).
\end{array} \tag{6}$$

These sets of coordinates, are, of course, not independent and linear relations between them can be found. Indeed, after some simple algebra we get

$$\begin{gathered}
\boldsymbol{r}_i = \boldsymbol{R} + \frac{m_j}{m_i + m_j}\boldsymbol{r}_{ij} - \frac{m_k}{m_i + m_j + m_k}\boldsymbol{q}_{ijk} - \frac{m_l}{M}\boldsymbol{\rho}_l, \\
\boldsymbol{r}_j = \boldsymbol{R} - \frac{m_i}{m_i + m_j}\boldsymbol{r}_{ij} - \frac{m_k}{m_i + m_j + m_k}\boldsymbol{q}_{ijk} - \frac{m_l}{M}\boldsymbol{\rho}_l, \\
\boldsymbol{r}_k = \boldsymbol{R} + \frac{m_i + m_j}{m_i + m_j + m_k}\boldsymbol{q}_{ijk} - \frac{m_l}{M}\boldsymbol{\rho}_l, \\
\boldsymbol{r}_l = \boldsymbol{R} + \frac{m_i + m_j + m_k}{M}\boldsymbol{\rho}_l,
\end{gathered} \tag{7}$$

where M is the total mass of the system. Thus, every one of the six Jacobi sets may be expressed as a function of the coordinates belonging to any one of them.

The momenta $\boldsymbol{P}, \boldsymbol{p}_{ij}, \boldsymbol{s}_{ijk}$ and $\boldsymbol{\pi}_l$, canonically conjugated to the coordinates $\boldsymbol{R}, \boldsymbol{r}_{ij}, \boldsymbol{q}_{ijk}$ and $\boldsymbol{\rho}_l$, respectively, may be obtained by one of the standard methods and they are

$$\begin{gathered}
\boldsymbol{P} = \boldsymbol{p}_i + \boldsymbol{p}_j + \boldsymbol{p}_k + \boldsymbol{p}_l, \\
\boldsymbol{p}_{ij} = (m_j\boldsymbol{p}_i - m_i\boldsymbol{p}_j)(m_i + m_j)^{-1}, \\
\boldsymbol{s}_{ijk} = [(m_i + m_j)\boldsymbol{p}_k - m_k(\boldsymbol{p}_i + \boldsymbol{p}_j)](m_i + m_j + m_k)^{-1}, \\
\boldsymbol{\pi}_l = [(m_i + m_j + m_k)\boldsymbol{p}_l - m_l(\boldsymbol{p}_i + \boldsymbol{p}_j + \boldsymbol{p}_k)]M^{-1}.
\end{gathered} \tag{8}$$

The three kinds of reduced masses associated to the relative particles come out by a simple inspection of Eqs (5) and (8). Their expressions are

$$m_{ij} = m_i m_j (m_i+m_j)^{-1}, \; m_{ijk} = (m_i+m_j)m_k(m_i+m_j+m_k)^{-1}$$

$$\mu_l = m_l(m_i+m_j+m_k)M^{-1}. \tag{9}$$

In any one of the six Jacobi frames the Schröndinger equation is obtained by the well-known canonical quantization rules. Thus in the case of a bound-state system of four distinguishable particles, with the CM-motion split off, this equation reads

$$(\alpha_{ij}\nabla^2_{r_{ij}}+\beta_{ijk}\nabla^2_{q_{ijk}}+\gamma_l\nabla^2_{\varrho}-K^2)\Psi = v_T\Psi. \tag{10}$$

Here,

$$v_T = \sum_{i<j} v_{ij}(r_{ij}), \;\; r_{ij} = |\boldsymbol{r}_{ij}|, \tag{11}$$

$$v_{ij} = M\hbar^{-2}V_{ij}(r_{ij}), \qquad K^2 = M\hbar^{-2}|E|, \qquad E = -|E|, \tag{12}$$

$$\alpha_{ij} = M/2m_{ij}, \qquad \beta_{ijk} = M/2m_{ijk}, \qquad \gamma_l = M/2\mu\,. \tag{13}$$

We notice that within any Jacobi set, as a result of restrictions $i<j$, $k<l$, the subscripts of $\boldsymbol{r}_{ij}$ fix the labels of the other two coordinates. For this we will always suppress the labels of $\boldsymbol{q}$ and $\boldsymbol{\rho}$ and those of the related quantities whenever no confusion may arise.

Now we adjust Eyges's approach to the dimensionality of our problem. Consider the coordinates $\boldsymbol{r}_{ij}, \boldsymbol{q}, \boldsymbol{\rho}$ which define the particle configuration in the CM-system. In other words, we introduce a nine-dimensional configuration space defined by any one of the six sets of internal Jacobi coordinates. A general point of this space will be represented by a nine-component vector $\boldsymbol{Q}$ so that in general Ψ will be a function of $\boldsymbol{Q}$. Each of the six equations included in (10) is equivalent to an integral equation for Ψ of the following form

$$\Psi(\boldsymbol{Q}) = -\int G(\boldsymbol{Q}-\boldsymbol{Q}')v_T(\boldsymbol{Q}')\Psi(\boldsymbol{Q}')d\boldsymbol{Q}', \tag{14}$$

where the Green's function G is the solution of the differential equation

$$(\alpha_{ij}\nabla^2_{r_{ij}}+\beta\nabla^2_{q}+\gamma\nabla^2_{\varrho}-K^2)G_{ij}(\boldsymbol{r}_{ij}-\boldsymbol{r}'_{ij}, \boldsymbol{q}-\boldsymbol{q}', \boldsymbol{\rho}-\boldsymbol{\rho}') =$$

$$= -\delta(\boldsymbol{r}_{ij}-\boldsymbol{r}'_{ij})\delta(\boldsymbol{q}-\boldsymbol{q}')\delta(\boldsymbol{\rho}-\boldsymbol{\rho}'), \tag{15}$$

and it may be represented in integral forms

$$G_{ij} = \frac{1}{(2\pi)^9}\int \frac{\exp\,[i\boldsymbol{k}(\boldsymbol{r}_{ij}-\boldsymbol{r}'_{ij})+i\boldsymbol{t}(\boldsymbol{q}-\boldsymbol{q}')+i\boldsymbol{\lambda}(\boldsymbol{\rho}-\boldsymbol{\rho}')]}{\alpha_{ij}k^2+\beta t^2+\gamma\lambda^2+K^2}\, d\boldsymbol{k}d\boldsymbol{t}d\boldsymbol{\lambda} \tag{16}$$

It is easy to show that all the Green's functions expressed in their proper Jacobi frames are identical, *i.e.*

$$G_{12}(\boldsymbol{r}_{12}-\boldsymbol{r}'_{12}, \boldsymbol{q}_{123}-\boldsymbol{q}'_{123}, \boldsymbol{\rho}_4-\boldsymbol{\rho}'_4) = G_{34}(\boldsymbol{r}_{34}-\boldsymbol{r}'_{34}, \boldsymbol{q}_{341}-\boldsymbol{q}'_{341}, \boldsymbol{\rho}_2-\boldsymbol{\rho}'_2),$$

and so on.

3. "Two-body orbital" decomposition of the wave function. The basic set of coupled integral equations

Following Eyges we write Ψ as a sum of six "two-body orbitals".

$$\Psi(Q) = \sum_{i<j} \psi_{ij}(Q), \tag{17}$$

where on the right-hand side for each "orbital" the proper Jacobi coordinates are used. Each of these "orbitals" is attached to one of the pair-potentials by the definition

$$\psi_{ij}(Q) = -\int G_{ij}(Q-Q')v_{ij}(Q')\Psi(Q')dQ' \tag{18}$$

in such a way that $\Psi(Q)$ given in (17) satisfies the Schrödinger equation (or the equivalent integral equation (14)). Now, substituting Eq. (17) in the right-hand side of Eq. (18) we get

$$\psi_{ij}(\boldsymbol{r}_{ij}, \boldsymbol{q}, \boldsymbol{\rho}) = -\int G_{ij}(\boldsymbol{r}_{ij}-\boldsymbol{r}'_{ij}, \boldsymbol{q}-\boldsymbol{q}'_{ijk}, \boldsymbol{\rho}-\boldsymbol{\rho}'_l)v_{ij}(r'_{ij})\times$$
$$\times \sum_{m<n} \psi_{mn}(\boldsymbol{r}'_{mn}, \boldsymbol{q}', \boldsymbol{\rho}')d\boldsymbol{r}'_{ij}d\boldsymbol{q}'_{ijk}d\boldsymbol{\rho}'_l, \tag{19}$$

where $i, j, k, l = 1, 2, 3, 4$ so that $i < j$ and $k < l$ simultaneously. Therefore, (19) represent a general set of six coupled integral equations which, if solved analytically, would give a complete solution of the four body problem. Every equation embodied in (19) couples each "two-body orbital" to itself and to the other five, while the integrations are performed with respect to Jacobi coordinates related to the corresponding two-body potential. Therefore, when passing to explicitate the right-hand side of Eq. (19) the "orbitals" $\psi_{mn}(\boldsymbol{r}'_{mn}, \boldsymbol{q}', \varrho')$ will be written as functions of $\boldsymbol{r}'_{ij}, \boldsymbol{q}'_{ijk}, \boldsymbol{\rho}'_l$. Thus, afer some tedious algebra we get

$$\psi_{ij}(\boldsymbol{r}_{ij}, \boldsymbol{q}, \boldsymbol{\rho}) = -\int G_{ij}(\boldsymbol{r}_{ij}-\boldsymbol{r}'_{ij}, \boldsymbol{q}-\boldsymbol{q}', \boldsymbol{\rho}-\boldsymbol{\rho}')v_{ij}(r'_{ij})[\psi_{ij}(\boldsymbol{r}'_{ii}, \boldsymbol{q}', \boldsymbol{\rho}')+$$
$$+\mathcal{S}_{ij}(r'_{ij}, \boldsymbol{q}', \boldsymbol{\rho}')]d\boldsymbol{r}'_{ij}d\boldsymbol{q}'d\boldsymbol{\rho}', \tag{20}$$

where the symbols $\mathcal{S}_{ij}$ are as follows

$$\mathcal{S}_{12} = \psi_{13}\left(\frac{m_{12}}{m_1}\boldsymbol{r}'_{12}-\boldsymbol{q}', -\frac{m_{12}}{m_{132}}\boldsymbol{r}'_{12}-\frac{m_{13}}{m_1}\boldsymbol{q}', \boldsymbol{\rho}'\right)+$$
$$+\psi_{23}\left(-\frac{m_{12}}{m_2}\boldsymbol{r}'_{12}-\boldsymbol{q}', \frac{m_{12}}{m_{231}}\boldsymbol{r}'_{12}-\frac{m_{23}}{m_2}\boldsymbol{q}', \boldsymbol{\rho}'\right)+$$
$$+\psi_{14}\left(\frac{m_{12}}{m_1}\boldsymbol{r}'_{12}-\frac{m_3m_4}{M\mu_4}\boldsymbol{q}'-\boldsymbol{\rho}', -\frac{m_{12}}{m_{142}}\boldsymbol{r}'_{12}-\frac{m_{14}m_3m_4}{m_1M\mu_4}\boldsymbol{q}'-\right.$$
$$\left.-\frac{m_{14}}{m_1}\boldsymbol{\rho}', \frac{m_{123}}{\mu_3}\boldsymbol{q}'-\frac{m_3m_4}{\mu_3M}\boldsymbol{\rho}'\right)+$$
$$+\psi_{24}\left(-\frac{m_{12}}{m_2}\boldsymbol{r}'_{12}-\frac{m_3m_4}{M\mu_4}\boldsymbol{q}'-\boldsymbol{\rho}', \frac{m_{12}}{m_{241}}\boldsymbol{r}'_{12}-\frac{m_{24}m_3m_4}{m_2M\mu_4}\boldsymbol{q}'-\right.$$
$$\left.-\frac{m_{24}}{m_2}\boldsymbol{\rho}', \frac{m_{123}}{\mu_3}\boldsymbol{q}'-\frac{m_3m_4}{\mu_3M}\boldsymbol{\rho}'\right)+$$

$$+\psi_{34}\left(\frac{m_{123}}{m_3}\boldsymbol{q}'-\boldsymbol{\rho}',\ \frac{m_{12}}{m_1}\boldsymbol{r}'_{12}-\frac{m_{34}}{\mu_4}\boldsymbol{q}'-\frac{m_{34}}{m_3}\boldsymbol{\rho}',-\frac{m_{12}}{\mu_2}\boldsymbol{r}'_1-\right.$$

$$\left.-\frac{m_2m_3m_4}{M\mu_2\mu_4}\boldsymbol{q}'-\frac{m_2m_4}{\mu_2M}\boldsymbol{\rho}'\right),$$

$$\mathscr{S}_{13}=\psi_{12}\left(\frac{m_{13}}{m_1}\boldsymbol{r}'_{13}-\boldsymbol{q}',-\frac{m_{13}}{m_{123}}\boldsymbol{r}'_{13}-\frac{m_{12}}{m_1}\boldsymbol{q}',\boldsymbol{\rho}'\right)+$$

$$+\psi_{23}\left(\frac{m_{13}}{m_3}\boldsymbol{r}'_{13}+\boldsymbol{q}',\ \frac{m_{13}}{m_{231}}\boldsymbol{r}'_{13}-\frac{m_{23}}{m_3}\boldsymbol{q}',\boldsymbol{\rho}'\right)+$$

$$+\psi_{14}\left(\frac{m_{13}}{m_1}\boldsymbol{r}'_{13}-\frac{m_2m_4}{M\mu_4}\boldsymbol{q}'-\boldsymbol{\rho}',-\frac{m_{13}}{m_1+m_4}\boldsymbol{r}'_{13}+\left(\frac{m_{132}}{m_2}+\frac{m_{14}m_2}{\mu_4M}\right)\boldsymbol{q}'-\right.$$

$$\left.-\frac{m_{14}}{m_1}\boldsymbol{\rho}',-\frac{m_{13}}{\mu_3}\boldsymbol{r}'_{13}-\frac{m_2m_3m_4}{M\mu_3\mu_4}\boldsymbol{q}'-\frac{m_3m_4}{\mu_2M}\boldsymbol{\rho}'\right)+$$

$$+\psi_{34}\left(-\frac{m_{13}}{m_3}\boldsymbol{r}'_{13}-\frac{m_2m_4}{M\mu_4}\boldsymbol{q}'-\boldsymbol{\rho}',\ \frac{m_{13}}{m_{341}}\boldsymbol{r}'_{13}-\frac{m_{34}m_2m_4}{m_3M\mu_4}\boldsymbol{q}'-\frac{m_{34}}{m_3}\boldsymbol{\rho}',\ \frac{m_{132}}{\mu_2}\boldsymbol{q}'-\frac{m_2m_4}{\mu_2M}\boldsymbol{\rho}'\right)+$$

$$+\psi_{24}\left(\frac{m_{132}}{m_2}\boldsymbol{q}'-\boldsymbol{\rho}',\ \frac{m_{13}}{m_1}\boldsymbol{r}'_{13}-\frac{m_{24}}{\mu_4}\boldsymbol{q}'-\frac{m_{24}}{m_2}\boldsymbol{\rho}',-\frac{m_{13}}{\mu_3}\boldsymbol{r}'_{13}-\frac{m_2m_3m_4}{M\mu_3\mu_4}\boldsymbol{q}'-\frac{m_3m_4}{\mu_3M}\boldsymbol{\rho}'\right),$$

$$\mathscr{S}_{14}=\psi_{24}\left(\frac{m_{14}}{m_4}\boldsymbol{r}'_{14}+\boldsymbol{q}',\ \frac{m_{14}}{m_{241}}\boldsymbol{r}'_{14}-\frac{m_{24}}{m_4}\boldsymbol{q}',\boldsymbol{\rho}'\right)+$$

$$+\psi_{12}\left(\frac{m_{14}}{m_1}\boldsymbol{r}'_{14}-\boldsymbol{q}',-\frac{m_{14}}{m_1+m_2}\boldsymbol{r}'_{14}-\frac{m_{12}m_4m_3}{m_1M\mu_3}\boldsymbol{q}'+\boldsymbol{\rho}',\right.$$

$$\left.-\frac{m_{14}}{\mu_4}\boldsymbol{r}'_{14}-\frac{m_2m_3m_4}{M\mu_3\mu_4}\boldsymbol{q}'-\frac{m_3m_4}{M\mu_4}\boldsymbol{\rho}'\right)+$$

$$+\psi_{13}\left(\frac{m_{14}}{m_1}\boldsymbol{r}'_{14}-\frac{m_2m_3}{\mu_3M}\boldsymbol{q}'-\boldsymbol{\rho}',-\frac{m_{14}}{m_1+m_3}\boldsymbol{r}'_{14}+\left(\frac{m_{142}}{m_2}+\frac{m_{13}m_2}{\mu_3M}\right)\boldsymbol{q}'-\right.$$

$$\left.-\frac{m_{13}}{m_1}\boldsymbol{\rho}',-\frac{m_{14}}{\mu_4}\boldsymbol{r}'_{14}-\frac{m_2m_3m_4}{M\mu_3\mu_4}\boldsymbol{q}'-\frac{m_3m_4}{M\mu_4}\boldsymbol{\rho}'\right)+$$

$$+\psi_{34}\left(\frac{m_{14}}{m_4}\boldsymbol{r}'_{14}+\frac{m_2m_3}{M\mu_3}\boldsymbol{q}'+\boldsymbol{\rho}',\ \frac{m_{14}}{m_{341}}\boldsymbol{r}'_{14}-\frac{m_{34}m_2m_3}{m_4M\mu_3}\boldsymbol{q}'-\frac{m_{34}}{m_4}\boldsymbol{\rho}',\right.$$

$$\left.\frac{m_{142}}{\mu_2}\boldsymbol{q}'-\frac{m_2m_3}{\mu_2M}\boldsymbol{\rho}'\right)+$$

$$+\psi_{23}\left(\frac{m_{142}}{m_2}\boldsymbol{q}'-\boldsymbol{\rho}',\ \frac{m_{14}}{m_1}\boldsymbol{r}'_{14}-\frac{m_{23}}{\mu_3}\boldsymbol{q}'-\frac{m_{23}}{m_2}\boldsymbol{\rho}',-\frac{m_{14}}{\mu_4}\boldsymbol{r}'_{14}-\right.$$

$$\left.-\frac{m_2m_3m_4}{M\mu_3\mu_4}\boldsymbol{q}'-\frac{m_3m_4}{M\mu_4}\boldsymbol{\rho}'\right),$$

$$\mathscr{S}_{23} = \psi_{12}\left(-\frac{m_{23}}{m_2}\boldsymbol{r}'_{23}+\boldsymbol{q}',\ -\frac{m_{23}}{m_{123}}\boldsymbol{r}'_{23}-\frac{m_{12}}{m_2}\boldsymbol{q}',\ \boldsymbol{\rho}'\right)+$$

$$+\psi_{13}\left(\frac{m_{23}}{m_3}\boldsymbol{r}'_{23}+\boldsymbol{q}',\ \frac{m_{23}}{m_{132}}\boldsymbol{r}'_{23}-\frac{m_{13}}{m_3}\boldsymbol{q}',\ \boldsymbol{\rho}'\right)+$$

$$+\psi_{24}\left(\frac{m_{23}}{m_2}\boldsymbol{r}'_{23}-\frac{m_1 m_4}{M\mu_4}\boldsymbol{q}'-\boldsymbol{\rho}',\ -\frac{m_{23}}{m_2+m_4}\boldsymbol{r}'_{23}+\left(\frac{m_{231}}{m_1}+\frac{m_{24}m_1}{\mu_4 M}\right)\boldsymbol{q}'-\right.$$

$$\left.-\frac{m_{24}}{m_2}\boldsymbol{\rho}',\ -\frac{m_{23}}{\mu_3}\boldsymbol{r}'_{23}-\frac{m_1 m_3 m_4}{M\mu_3\mu_4}\boldsymbol{q}'-\frac{m_3 m_4}{\mu_3 M}\boldsymbol{\rho}'\right)+$$

$$+\psi_{34}\left(-\frac{m_{23}}{m_3}\boldsymbol{r}'_{23}-\frac{m_1 m_4}{M\mu_4}\boldsymbol{q}'-\boldsymbol{\rho}',\ \frac{m_{23}}{m_3+m_4}\boldsymbol{r}'_{23}+\left(\frac{m_{231}}{m_1}+\frac{m_{34}m_1}{\mu_4 M}\right)\boldsymbol{q}'-\right.$$

$$\left.-\frac{m_{34}}{m_3}\boldsymbol{\rho}',\ \frac{m_{23}}{\mu_2}\boldsymbol{r}'_{23}-\frac{m_1 m_2 m_4}{M\mu_2\mu_4}\boldsymbol{q}'-\frac{m_2 m_4}{\mu_2 M}\boldsymbol{\rho}'\right)+$$

$$+\psi_{14}\left(\frac{m_{231}}{m_1}\boldsymbol{q}'-\boldsymbol{\rho}',\ \frac{m_{23}}{m_2}\boldsymbol{r}'_{23}-\frac{m_{14}}{\mu_4}\boldsymbol{q}'-\frac{m_{14}}{m_1}\boldsymbol{\rho}',\ -\frac{m_{23}}{\mu_3}\boldsymbol{r}'_{23}-\right.$$

$$\left.-\frac{m_1 m_3 m_4}{M\mu_3\mu_4}\boldsymbol{q}'-\frac{m_3 m_4}{\mu_3 M}\boldsymbol{\rho}'\right),$$

$$\mathscr{S}_{24} = \psi_{14}\left(\frac{m_{24}}{m_4}\boldsymbol{r}'_{24}+\boldsymbol{q}',\ \frac{m_{24}}{m_{142}}\boldsymbol{r}'_{24}-\frac{m_{14}}{m_4}\boldsymbol{q}',\ \boldsymbol{\rho}'\right)+$$

$$+\psi_{12}\left(-\frac{m_{24}}{m_2}\boldsymbol{r}'_{24}+\boldsymbol{q}',\ -\frac{m_{24}}{m_1+m_2}\boldsymbol{r}'_{24}-\frac{m_{12}m_3m_4}{Mm_2\mu_3}\boldsymbol{q}'+\boldsymbol{\rho}',\right.$$

$$\left.-\frac{m_{24}}{\mu_4}\boldsymbol{r}'_{24}-\frac{m_1 m_3 m_4}{M\mu_3\mu_4}\boldsymbol{q}'-\frac{m_3 m_4}{M\mu_4}\boldsymbol{\rho}'\right)+$$

$$+\psi_{23}\left(\frac{m_{24}}{m_2}\boldsymbol{r}'_{24}-\frac{m_1 m_3}{M\mu_3}\boldsymbol{q}'-\boldsymbol{\rho}',\ -\frac{m_{24}}{m_2+m_3}\boldsymbol{r}'_{24}+\left(\frac{m_{241}}{m_1}+\frac{m_{23}m_1}{M\mu_3}\right)\boldsymbol{q}'-\right.$$

$$\left.-\frac{m_{23}}{m_2}\boldsymbol{\rho}',\ -\frac{m_{24}}{\mu_4}\boldsymbol{r}'_{24}-\frac{m_1 m_3 m_4}{M\mu_3\mu_4}\boldsymbol{q}'-\frac{m_3 m_4}{M\mu_4}\boldsymbol{\rho}'\right)+$$

$$+\psi_{34}\left(\frac{m_{24}}{m_4}\boldsymbol{r}'_{24}+\frac{m_1 m_3}{M\mu_3}\boldsymbol{q}'+\boldsymbol{\rho}',\ \frac{m_{24}}{m_3+m_4}\boldsymbol{r}'_{24}+\left(\frac{m_{241}}{m_1}+\frac{m_{34}m_1}{M\mu_3}\right)\boldsymbol{q}'-\right.$$

$$\left.-\frac{m_{34}}{m_4}\boldsymbol{\rho}',\ \frac{m_{24}}{\mu_2}\boldsymbol{r}'_{24}-\frac{m_1 m_2 m_3}{M\mu_2\mu_3}\boldsymbol{q}'-\frac{m_2 m_3}{M\mu_2}\boldsymbol{\rho}'\right)+$$

$$+\psi_{13}\left(\frac{m_{241}}{m_1}\boldsymbol{q}'-\boldsymbol{\rho}',\ \frac{m_{24}}{m_2}\boldsymbol{r}'_{24}-\frac{m_{13}}{\mu_3}\boldsymbol{q}'-\frac{m_{13}}{m_1}\boldsymbol{\rho}',\ -\frac{m_{24}}{\mu_4}\boldsymbol{r}'_{24}-\right.$$

$$\left.-\frac{m_1 m_3 m_4}{M\mu_3\mu_4}\boldsymbol{q}'-\frac{m_3 m_4}{M\mu_4}\boldsymbol{\rho}'\right),$$

$$\mathscr{S}_{34} = \psi_{13}\left(-\frac{m_{34}}{m_3}\boldsymbol{r}'_{34}+\boldsymbol{q}',\ -\frac{m_{34}}{m_1+m_3}\boldsymbol{r}'_{34}-\frac{m_{13}m_2m_4}{m_3\mu_2M}\boldsymbol{q}'+\boldsymbol{\rho}',\right.$$

$$\left.-\frac{m_{34}}{\mu_4}\boldsymbol{r}'_{34}-\frac{m_1m_2m_4}{M\mu_2\mu_4}\boldsymbol{q}'-\frac{m_2m_4}{M\mu_4}\boldsymbol{\rho}'\right)+$$

$$+\psi_{14}\left(\frac{m_{34}}{m_4}\boldsymbol{r}'_{34}+\boldsymbol{q}',\ \frac{m_{34}}{m_1+m_4}\boldsymbol{r}'_{34}-\frac{m_{14}m_2m_3}{m_4\mu_2M}\boldsymbol{q}'+\boldsymbol{\rho}',\ \frac{m_{34}}{\mu_3}\boldsymbol{r}'_{34}-\right.$$

$$\left.-\frac{m_1m_2m_3}{M\,\mu_2\mu_3}\boldsymbol{q}'-\frac{m_2m_3}{M\mu_3}\boldsymbol{\rho}'\right)+$$

$$+\psi_{23}\left(-\frac{m_{34}}{m_3}\boldsymbol{r}'_{34}+\frac{m_1m_2}{M\,\mu_2}\boldsymbol{q}'+\boldsymbol{\rho}',-\frac{m_{34}}{m_2+m_3}\boldsymbol{r}'_{34}+\left(\frac{m_{341}}{m_1}+\frac{m_{23}m_1}{M\mu_2}\right)\boldsymbol{q}'-\right.$$

$$\left.-\frac{m_{23}}{m_3}\boldsymbol{\rho}',-\frac{m_{34}}{\mu_4}\boldsymbol{r}'_{34}-\frac{m_1m_2m_4}{M\mu_2\mu_4}\boldsymbol{q}'-\frac{m_2m_4}{M\mu_4}\boldsymbol{\rho}'\right)+$$

$$+\psi_{24}\left(\frac{m_{34}}{m_4}\boldsymbol{r}'_{34}+\frac{m_1m_2}{M\mu_2}\boldsymbol{q}'+\boldsymbol{\rho}',\ \frac{m_{34}}{m_2+m_4}\boldsymbol{r}'_{34}+\left(\frac{m_{341}}{m_1}+\frac{m_{24}m_1}{M\mu_2}\right)\boldsymbol{q}'-\right.$$

$$\left.-\frac{m_{24}}{m_4}\boldsymbol{\rho}',\frac{m_{34}}{\mu_3}\boldsymbol{r}'_{34}-\frac{m_1m_2m_3}{M\mu_2\mu_3}\boldsymbol{q}'-\frac{m_2m_3}{M\mu_3}\boldsymbol{\rho}'\right)+$$

$$+\psi_{12}\left(\frac{m_{341}}{m_1}\boldsymbol{q}'-\boldsymbol{\rho}',\ \frac{m_{34}}{m_3}\boldsymbol{r}'_{34}-\frac{m_{12}}{\mu_2}\boldsymbol{q}'-\frac{m_{12}}{m_1}\boldsymbol{\rho}',\ -\frac{m_{34}}{\mu_4}\boldsymbol{r}'_{34}-\right.$$

$$\left.-\frac{m_1m_2m_4}{M\mu_2\mu_4}\boldsymbol{q}'-\frac{m_2m_4}{M\mu_4}\boldsymbol{\rho}'\right).$$

As has been emphasized in the three-particle case, a convenient way to simplify the set of integral equations, where some of the functions appear evaluated at transformed arguments, is to use the Fourier transforms of the "orbitals" and re-express the four-body problem in a nine-dimensional momentum space. We define the Fourier transforms of the "orbitals" by the relations

$$\psi_{ij}(\boldsymbol{r}_{ij},\boldsymbol{q},\boldsymbol{\rho}) = (2\pi)^{-\frac{9}{2}}\int \Phi_{ij}(\boldsymbol{k},\ \boldsymbol{t},\ \boldsymbol{\lambda})e^{i(\boldsymbol{r}_{ij}\boldsymbol{k}+\boldsymbol{q}\boldsymbol{t}+\boldsymbol{\rho}\boldsymbol{\lambda})}d\boldsymbol{k}d\boldsymbol{t}d\boldsymbol{\lambda},$$

$$\Phi_{ij}(\boldsymbol{k},\ \boldsymbol{t},\boldsymbol{\lambda}) = (2\pi)^{-\frac{9}{2}}\int \psi_{ij}(\boldsymbol{r}_{ij},\boldsymbol{q},\ \boldsymbol{\rho})e^{-i(\boldsymbol{k}\boldsymbol{r}_{ij}+\boldsymbol{t}\boldsymbol{q}+\boldsymbol{\lambda}\boldsymbol{\rho})}d\boldsymbol{r}d\boldsymbol{q}d\boldsymbol{\rho}.$$

Now, in the right-hand side of Eq. (20) the Fourier transforms of the "orbitals" and the integral representation of G_{ij} are substituted. Then, performing the Fourier transformation of the whole equation and using some standard tricks, the following equation emerges

$$\Phi_{ij}(\boldsymbol{k},\ \boldsymbol{t},\ \boldsymbol{\lambda}) = -\frac{(2\pi)^{-3}}{\alpha_{ij}k^2+\beta t^2+\gamma\lambda^2+K^2}\int v_{ij}(r_{ij})\ [\Phi_{ij}(\boldsymbol{k}',\ \boldsymbol{t},\ \boldsymbol{\lambda})+$$

$$\mathscr{T}_{ij}(\boldsymbol{k}',\ \boldsymbol{t},\ \boldsymbol{\lambda})]\ e^{i(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{r}_{ij}}\,d\boldsymbol{r}_{ij}\,d\boldsymbol{k}', \tag{21}$$

where $\mathscr{T}_{ij}(\boldsymbol{k}',\ \boldsymbol{t},\ \boldsymbol{\lambda})$ is the Fourier transform of $\mathscr{S}_{ij}(\boldsymbol{r}'_{ij},\boldsymbol{q}',\ \boldsymbol{\rho}')$, that is

$$\mathscr{T}_{ij}(\boldsymbol{k}',\ \boldsymbol{t},\ \lambda) = 2\pi)^{-\frac{9}{2}}\int \mathscr{S}_{ij}(\boldsymbol{r}'_{ij},\boldsymbol{q}',\ \boldsymbol{\rho}')e^{-i(\boldsymbol{k}'\boldsymbol{r}'_{ij}+\boldsymbol{t}\boldsymbol{q}'+\boldsymbol{\lambda}\boldsymbol{\rho})}d\boldsymbol{r}'_{ij}d\boldsymbol{q}'d\varrho'. \tag{22}$$

Substituting here the expressions of $\mathscr{S}_{ij}$ and the Fourier transforms of the involved "orbitals", after some calculations the functions $\mathscr{T}_{ij}$ as a sum of the corresponding five "momentum orbitals" are obtained. For example,

$$\mathscr{T}_{12} = \Phi_{13}\left(\frac{m_{13}}{m_1}\boldsymbol{k}' - \frac{m_{13}}{m_{123}}\boldsymbol{t}, -\boldsymbol{k}' - \frac{m_{12}}{m_1}\boldsymbol{t}, \boldsymbol{\lambda}\right) +$$

$$+\Phi_{23}\left(-\frac{m_{23}}{m_2}\boldsymbol{k}' - \frac{m_{23}}{m_{123}}\boldsymbol{t}, \boldsymbol{k}' - \frac{m_{12}}{m_2}\boldsymbol{t}, \boldsymbol{\lambda}\right) +$$

$$+\Phi_{14}\left(\frac{m_{14}}{m_1}\boldsymbol{k}' - \frac{m_{14}}{m_1+m_2}\boldsymbol{t} - \frac{m_{14}}{\mu_4}\boldsymbol{\lambda}, -\boldsymbol{k}' - \frac{m_{12}m_3m_4}{m_1\mu_3 M}\boldsymbol{t} - \right.$$

$$\left. - \frac{m_2m_3m_4}{M\mu_3\mu_4}\boldsymbol{\lambda}, \boldsymbol{t} - \frac{m_3m_4}{M\mu_4}\boldsymbol{\lambda}\right) +$$

$$+\Phi_{24}\left(-\frac{m_{24}}{m_2}\boldsymbol{k}' - \frac{m_{24}}{m_1+m_2}\boldsymbol{t} - \frac{m_{24}}{\mu_4}\boldsymbol{\lambda}, \boldsymbol{k}' - \frac{m_{12}m_3m_4}{m_2\mu_3 M}\boldsymbol{t} - \right.$$

$$\left. - \frac{m_1m_3m_4}{M\mu_3\mu_4}\lambda, \boldsymbol{t} - \frac{m_3m_4}{M\mu_4}\boldsymbol{\lambda}\right) +$$

$$+\Phi_{34}\left(\frac{m_{34}}{m_3}\boldsymbol{t} - \frac{m_{34}}{\mu_4}\boldsymbol{\lambda}, \frac{m_{341}}{m_1}\boldsymbol{k}' - \frac{m_{12}}{\mu_2}\boldsymbol{t} - \frac{m_1m_2m_4}{M\mu_2\mu_4}\boldsymbol{\lambda}, \right.$$

$$\left. -\boldsymbol{k}' - \frac{m_{12}}{m_1}\boldsymbol{t} - \frac{m_2m_4}{M\mu_4}\right).$$

Now, with the aid of our index-convention the six coupled equations of the system (21) result directly. They represent the basic set of integral equations for the "mementum orbitals" in the case of four distinguishable particles interacting with each other by two-body forces. Concerning this system there is a remarkable aspect of the formalism to be emphasized. As the number of particles is increased, the number of coupled equations also increases; for N distinguishable particles this number is $N(N-1)/2$. But, and this is an essential thing, the dimensions of the involved integrals remain unchanged: they are to be performed with respect to six "scalar" variables, just as in the three-body case.

In the cases when identical particles are present, the system (21) simplifies considerably, due to the symmetry requirements. Such cases of an obvious interest will be treated in a forthcoming paper.

4. *The angular momentum operator*

We end this work making some considerations about the total angular momentum operator of the system of four distinguishable particles.

Having in mind that the separation of the CM-motion was performed by means of a step by step reduction to the center-of-mass of two particles, the kinetic energy term, transformed from the cartesian frame to any Jacobi frame, looks

$$\sum_{n=1}^{4}\frac{p_n^2}{2m_n} = \frac{P^2}{2M} + \frac{p_{ij}^2}{2m_{ij}} + \frac{s^2}{2m} + \frac{\pi^2}{2\mu}.$$

In fact, this form of the kinetic energy was used in writing the Schrödinger equation (10). For the same reason the following relation holds for the total angular momentum of the system

$$\boldsymbol{L} = \sum_{n=1}^{4} (\boldsymbol{r}_n \times \boldsymbol{p}_n) = \boldsymbol{R} \times \boldsymbol{P} + \boldsymbol{r}_{ij} \times \boldsymbol{p}_{ij} + \boldsymbol{q} \times \boldsymbol{s} + \boldsymbol{\rho} \times \boldsymbol{\pi}. \tag{23}$$

Thus, after the canonical quantization in the $\{\boldsymbol{R}, \boldsymbol{r}_{ij}, \boldsymbol{q}, \boldsymbol{\rho}\}$ representation has been performed, it will hold, for instance, that

$$L_z = \frac{\hbar}{i} \sum_{n=1}^{4} \left(x_n \frac{\partial}{\partial y_n} - y_n \frac{\partial}{\partial x_n} \right) = \frac{\hbar}{i} \left[\left(R_x \frac{\partial}{\partial R_y} - R_y \frac{\partial}{\partial R_x} \right) + \right.$$

$$\left. + \left(x_{ij} \frac{\partial}{\partial y_{ij}} - y_{ij} \frac{\partial}{\partial x_{ij}} \right) + \left(q_x \frac{\partial}{\partial q_y} - q_y \frac{\partial}{\partial q_x} \right) + \left(\varrho_x \frac{\partial}{\partial \varrho_y} - \varrho_y \frac{\partial}{\partial \varrho_x} \right) \right],$$

where $\boldsymbol{R} = (R_x, R_y, R_z)$, $\boldsymbol{r}_{ij} = (x_{ij}, y_{ji}, z_{ij})$, etc.

Obviously, the cartesian frame expression of the operator $\boldsymbol{L}$ and its expressions in any Jacobi frame are identical, *i.e.*

$$\boldsymbol{L} = \boldsymbol{L}_{12} = \boldsymbol{L}_{13} = \boldsymbol{L}_{14} = \boldsymbol{L}_{23} = \boldsymbol{L}_{24} = \boldsymbol{L}_{34}.$$

It is easy to show that when requiring $\Psi(\boldsymbol{r}_{ij}, \boldsymbol{q}, \boldsymbol{\rho})$ to be a common eigenstate of L^2 and L_z, then similarly to the three-body case, each individual "orbital" is also a common eigenstate of L^2 and L_z. Indeed, according to the above remarks,

$$L^2\Psi = L_{12}^2 \psi_{12}(\boldsymbol{r}_{12}, \boldsymbol{q}_{123}, \boldsymbol{\rho}_4) + \ldots + L_{34}^2 \psi_{34}(\boldsymbol{r}_{34}, \boldsymbol{q}_{341}, \boldsymbol{\rho}_2), \tag{24}$$

and similarly for $L_z\Psi$. It is seen that each "orbital" must be also a common eigenstate of L^2 and L_z.

The above considerations are true in the N-particle case, too, since similarly to (23) in any Jacobi frame

$$\sum_{n=1}^{N} (\boldsymbol{r}_n \times \boldsymbol{p}_n) = \boldsymbol{R} \times \boldsymbol{P} + \boldsymbol{r}_{i_1 i_2} \times \boldsymbol{p}_{i'i_2} + \sum_{k=3}^{N} (\boldsymbol{q}_{i_1 i_2 \ldots ik} \times \boldsymbol{s}_{1122\ldots ik})$$

where $\boldsymbol{P}, \boldsymbol{p}_{i_1 i_2}$ and $\boldsymbol{s}_{i_1 i_2 \ldots ik}$ are the momente conjugated to $\boldsymbol{R}, \boldsymbol{r}_{i_1 i_2}$ and $\boldsymbol{q}_{i_1 i_2 \ldots ik}$, respectively.

APPENDIX

The proof is an elementary arithmetic argument. Indeed, by imposing successively the restrictions stated in (4), these remove a number of Jacobi sets as follows:

$i_1 < i_2$ remove $(N-2)!\, C_N^2 = \dfrac{N!}{2}$ sets,

$i_3 < i_4$ remove $\dfrac{1(N-2)!}{2!} C_N^2 = \dfrac{1}{2!} \dfrac{N!}{2}$ sets,

$$i_4 < i_5 \quad \text{remove} \quad \frac{2(N-2)!}{3!} C_N^2 = \frac{2}{3!}\frac{N!}{2} \quad \text{sets,}$$

$$i_5 < i_6 \quad \text{remove} \quad \frac{3(N-2)!}{4!} C_N^2 = \frac{3}{4!}\frac{N!}{2} \quad \text{sets,}$$

$$i_{N-1} < i_N \quad \text{remove} \quad \frac{(N-3)(N-2)!}{(N-2)!} C_N^2 = \frac{N-3}{(N-2)!}\frac{N!}{2} \quad \text{sets.}$$

Thus, the number of retained Jacobi sets is

$$\frac{N!}{2}\left[1-\left(\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{N-3}{(N-2)!}\right)\right].$$

It is easy to show by induction that the expression in the square brackets is equal to $1/(N-2)!$ and, hence, the transparent rule (4) indeed retains the necessary number of $N(N-1)/2$ Jacobi sets.

REFERENCES

[1] L. Eyges, *Phys. Rev.*, **121**, 1744 (1961) and *J. Math. Phys.*, **6**, 1320 (1965).
[2] J. R. Jasperse, M. H. Friedman, *Phys. Rev.*, **159**, 69 (1967).
[3] A. Messiah, *Quantum Mechanics*, North-Holland Publ. Co., Amsterdam 1961, Vol. I, Ch. IX, § 13.