QUANTUM MARKOVIAN PROCESSES IN EUCLIDEAN SPACE AND THE ELECTROMAGNETIC STRUCTURE OF THE SCHRÖDINGER EQUATIONS

By W. Garczyński

Institute of Theoretical Physics, University of Wrocław*

(Received December 20, 1969)

Quantum Markovian processes, introduced earlier by the author, in three dimensional Euclidean space are discussed. They correspond to the simplest possible quantum systems, *i.e.* to spinless particles moving under the influence of an external electromagnetic field and other possible forces. The most general Schrödinger equations are derived and discussed. In particular, the dependence of the coefficients of these equations on electromagnetic potentials is investigated without any reference to the analogy with the classical mechanics case.

1. *Introduction*

Physicists have witnessed various attempts to incorporate quantum mechanics into the theory of probabilities. A formal analogy between the Schrödinger equation and the Planck-Fokker equation in the theory of diffusion or with the heat conduction equation was noticed early [1]. On the other hand, the fundamental differences between these theories have also been realized [2]. In the quantum theory one deals with a complex valued probability amplitude instead of a real probability distribution as in the theory of diffusion. Moreover, the Schrödinger equation describes processes reversible in time while the Planck-Fokker equation concerns strictly irreversible processes. The "diffusion coefficient" in the Schrödinger equation is purely imaginary, in contrast to the diffusion theory, where this coefficient is reall.

In spite of these fundamental obstacles there were many, however unsatisfactory, attempts to relate quantum mechanics to the classical theory of Markovian stochastic processes [3], [4].

A new trend of research was initiated by the Feynman space-time approach to quantum mechanics, based on its heuristically defined path integral [5]. Many successful applications of this approach [6] have given rise to a need for the development of an adequate mathematical scheme which would make it rigorous. Such a scheme was proposed recently by the

* Address: Instytut Fizyki Teoretycznej UW, Wrocław 2, Cybulskiego 36, Polska.

author [7], [8], [9] and is based on the notion of quantum Markovian process of diffusional type.

The main idea of our approach is that to any quantum system there corresponds a quantum Markovian process in some set of states. The quantum Markovian process in the most general case is given by a set of amplitudes $a_{ik}(s, t)$ giving the probabilities $p_{ik}(s, t) = |a_{ik}(s, t)|^2$ of finding a system in the state k at the time t if it is known that at the time $s < t$ the state i out of the set $\mathscr{X}$ was occupied, these amplitudes are subjected to the following basic conditions

$$a_{ik}(s, t) = a^*_{ki}(t, s) \tag{i}$$

(motion reversibility condition)

$$\lim_{t \downarrow s} a_{ik}(s, t) = \delta_{ik} \tag{ii}$$

(time continuity property)

$$\sum_{j \in \mathscr{X}} a_{ij}(s, t)\, a^*_{kj}(s, \tau) = \delta_{ik} \tag{iii}$$

(unitarity requirement)

$$\sum_{j \in \mathscr{X}} a_{ij}(s, \tau)\, a_{jk}(\tau, t) = a_{ik}(s, t), \quad s < \tau < t. \tag{iv}$$

(quantum causality condition).

The Shrödinger equations follow from the last postulate, if some limits exist, in the same way as the Kolmogorov equations follow in the classical Markovian processes theory [10]. The idea of quantum Markovian processes underlies the whole quantum theory, including a theory of quantum fields and the S-matrix theory. We have arrived at it after an analysis of some criterions for the trivial nature of the axiomatic quantum field theory and especially the Haag theorem [11], [12]. Later, we found that some other scholars have approached this notion from different points of departure. However, they did not recognize its significance for quantum physics and did not develop it further. We give a list of corresponding references without any certainty that it is already complete [13], [14], [15], [16].

The second section of the paper is devoted to a brief account of our results concerning the theory of quantum Markovian processes in three dimensional Euclidean space of states of a one-particle quantum system. In the third section we derive an electromagnetic structure of Schrödinger equations, *i. e.* we establish the dependence of coefficients in these equations on the electromagnetic potentials of an external field.

2. *Basic postulates and Schrödinger equations*

A quantum Markovian process which corresponds to a spinless particle enclosed in the domain $\mathscr{X} \subseteq \mathscr{R}^3$ is characterized by the following postulates:

$$(s, \boldsymbol{y}; t, \boldsymbol{x}) = (t, \boldsymbol{x}; s, \boldsymbol{y})^*, \tag{i}$$

$$\lim_{t \downarrow s} \int_{\mathscr{X}} d\boldsymbol{y} \int_{\mathscr{X}} d\boldsymbol{x}\, f^*(\boldsymbol{y})\, (s, \boldsymbol{y}; t, \boldsymbol{x})\, g(\boldsymbol{x}) = \int_{\mathscr{X}} d\boldsymbol{x}\, f^*(\boldsymbol{x})\, g(\boldsymbol{x}) \tag{ii}$$

for any functions f and g from the space $L^2(\mathscr{X})$,

$$\int_{\mathscr{X}} d\boldsymbol{z}(s, \boldsymbol{y}; \tau, \boldsymbol{z})\,(s, \boldsymbol{x}; \tau, \boldsymbol{z})^* = \delta(\boldsymbol{y}-\boldsymbol{x}), \tag{iii}$$

$$\int_{\mathscr{X}} d\boldsymbol{z}\,(s, \boldsymbol{y}; \tau, \boldsymbol{z})\,(\tau, \boldsymbol{z}; t, \boldsymbol{x}) = (s, \boldsymbol{y}; t, \boldsymbol{x}), \qquad s < \tau < t, \tag{iv}$$

$$\lim_{\boldsymbol{y}\to\boldsymbol{y}'} \int_{\mathscr{X}} d\boldsymbol{x} f(\boldsymbol{x})\,(s, \boldsymbol{y}; t, \boldsymbol{x}) = \int_{\mathscr{X}} d\boldsymbol{x} f(\boldsymbol{x})\,(s, \boldsymbol{y}'; t, \boldsymbol{x}) \tag{v}$$

for any continuous and bounded function f.

We call these properties of the transition density amplitude $(s, \boldsymbol{y}; t, \boldsymbol{x})$ from the point $\boldsymbol{y}$ at the time s to the point $\boldsymbol{x}$ at the time $t > s$, respectively, motion reversibility condition, time-continuity condition, unitarity requirement, quantum causality condition and space--continuity condition.

In order to show it clear by how big is the variety of processes satisfying (i—v), we look at the family of operators associated naturally with them:

$$W(s, t)\, f(\boldsymbol{y}) = \begin{cases} \int_{\mathscr{X}} \mathrm{d}\boldsymbol{x}(s, \boldsymbol{y}; t, \boldsymbol{x}) f(\boldsymbol{x}) & \text{for } s < t, \\ f(\boldsymbol{y}) & \text{for } s = t \end{cases} \tag{1.2}$$

where f is any element of Hilbert space $L^2(\mathscr{X})$.

It is not difficult to deduce from the axioms (i—v) that $W(s, t)$ transform $L^2(\mathscr{X})$ back into the whole $L^2(\mathscr{X})$ again, that they are unitary operators strongly continuous at the point $t = s$, and that following formulae are valid:

$$W^+(s, t) = W^{-1}(s, t) = W(t, s)$$

$$W(s, \tau)\, W(\tau, t) = W(s, t). \tag{2.2}$$

In reverse, to any such family of operators there corresponds a quantum Markovian process given by

$$(s, \boldsymbol{y}; t, \boldsymbol{x}) = \sum_k \varphi_k^*(\boldsymbol{y})\, W(s, t)\, \varphi_k(\boldsymbol{x}) \tag{3.2}$$

where $\{\varphi_k\}$ is any complete and orthogonal set of functions in $L^2(\mathscr{X})$. For instance, if we take a family of operators $W(s, t)$ produced by a selfadjoint operator A

$$W(s, t) = \exp\{i(s-t)\, A\} \tag{4.2}$$

then we get, using for $\{\varphi_k\}$ the complete set of eigenfunctions of A,

$$(s, \boldsymbol{y}; t, \boldsymbol{x}) = \sum_k \varphi_k^*(\boldsymbol{y})\, \varphi_k(\boldsymbol{x}) \exp\{i(t-s)\lambda_k\}$$

$$A\varphi_k(\boldsymbol{x}) = \lambda_k \varphi_k(\boldsymbol{x}). \tag{5.2}$$

This is a well known formula for the Green function of the equation

$$\left[i \cdot \frac{\partial}{\partial t} - A(\boldsymbol{x})\right] (s, \boldsymbol{y}; t, \boldsymbol{x}) = 0. \tag{6.2}$$

Concluding, we have the one to one correspondence between the quantum Markovian processes and the families of unitary operators having the mentioned properties. It is clear from the last example that there are many processes satisfying (i—v).

The most interesting subclass of the quantum Markovian processes is formed by what we call quantum diffusion processes. They are characterized by the existence of the following limits:

A. $$\lim_{t\downarrow s} (t-s)^{-1} W(s, t)\, \Delta_k(\boldsymbol{y}) = a_k(s, \boldsymbol{y})$$

B. $$\lim_{t\downarrow s} (t-s)^{-1} W(s, t)\, \Delta_k \Delta_j(\boldsymbol{y}) = b_{kj}(s, \boldsymbol{y})$$

C. $$\lim_{t\downarrow s} (t-s)^{-1} [W(s, t) \cdot 1(\boldsymbol{y}) - 1] = c(s, \boldsymbol{y})$$

D. $$\lim_{t\downarrow s} (t-s)^{-1} W(s, t)\, \Delta_1^{n_1} \Delta_2^{n_2} \Delta_3^{n_3} \varphi(\boldsymbol{y}) = 0$$

for $n_1+n_2+n_3 = 3$ and for any bounded and continuous function $\varphi(\boldsymbol{x})$ and $\Delta_k(\boldsymbol{x}) = x_k - y_k$, $k = 1, 2, 3$.

The best known example of quantum Markovian process of diffusional type is the quantum Brownian motion process,

$$(s, \boldsymbol{y}; t, \boldsymbol{x}) = \left[\frac{m}{2\pi i \hbar (t-s)}\right]^{3/2} \cdot \exp\left\{i \frac{m}{2\hbar(t-s)} (\boldsymbol{y}-\boldsymbol{x})^2\right\}. \tag{7.2}$$

It describes the free motion of a particle with mass m in the space $\mathscr{R}^3$. We shall call it also the Feynman process. It is easy to calculate the coefficients a_k, b_{kj} and c from this amplitude, *viz.*,

$$a_k(s, \boldsymbol{y}) = 0$$

$$b_{kj}(s, \boldsymbol{y}) = i \frac{\hbar}{m} \cdot \delta_{kj}$$

$$c(s, \boldsymbol{y}) = 0. \tag{8.2}$$

The condition D is also satisfied here.

The conditions A—D permit us to derive the following differential equations from the causality postulate:

$$[\partial_s + K(s, \boldsymbol{y})]\, (s, \boldsymbol{y}; t, \boldsymbol{x})$$

$$\equiv \left[\partial_s + \frac{1}{2} b_{kj}(s, \boldsymbol{y})\, \partial_k \partial_j + a_k(s, \boldsymbol{y})\, \partial_k + c(s, \boldsymbol{y})\right] (s, \boldsymbol{y}; t, \boldsymbol{x}) = 0, \tag{9.2}$$

$$[-\partial_t + L(t, \boldsymbol{x}])\, (s, \boldsymbol{y}; t, \boldsymbol{x})$$

$$\equiv \left[-\partial_t + \frac{1}{2} \partial_k \partial_j b_{kj}(t, \boldsymbol{x}) - \partial_k a_k(t, \boldsymbol{x}) + c(t, \boldsymbol{x})\right] (s, \boldsymbol{y}; t, \boldsymbol{x}) = 0. \tag{10.2}$$

These equations coincide formally with the well known Kolmogorov equations in classical Markovian processes theory. Just to get an idea of how these equations follow from (iv), we notice that we have at $\tau = s$

$$(s-\Delta s, \boldsymbol{y}; t, \boldsymbol{x}) = \int_{\mathscr{X}} d\boldsymbol{z}(s-\Delta s, \boldsymbol{y}; s, \boldsymbol{z}) (s, \boldsymbol{z}; t, \boldsymbol{x}).$$

Expanding the amplitude $(s, \boldsymbol{z}; t, \boldsymbol{x})$ according to the Taylor formula at the point $\boldsymbol{z} = \boldsymbol{y}$ we get

$$(s-\Delta s, \boldsymbol{y}; t, \boldsymbol{x}) = [W(s-\Delta s, t) \cdot 1(\boldsymbol{y}) + W(s-\Delta s, t)\, \Delta_k(\boldsymbol{y})\, \partial_k +$$
$$+ \frac{1}{2} W(s-\Delta s, t)\, \Delta_k \Delta_j(\boldsymbol{y}) \partial_k \partial_j + R]\, (s, \boldsymbol{y}; t, \boldsymbol{x}).$$

Then, using A—D, we get in the limit $\Delta s \downarrow 0$

$$\lim_{\Delta s \downarrow 0} (\Delta s)^{-1} [(s-\Delta s, \boldsymbol{y}; t, \boldsymbol{x}) - (s, \boldsymbol{y}; t, \boldsymbol{x})] = K(s, \boldsymbol{y})\, (s, \boldsymbol{y}; t, \boldsymbol{x})$$

which is the first equation. A slightly more complicated method has to be applied in the derivation of the second equation.

The phenomenological coefficients a_k, b_{kj} and c are not quite arbitrary. Namely, we get from the reversibility axiom (i) by differentiation over s the relation

$$-K(s, \boldsymbol{y})\, (s, \boldsymbol{y}; t, \boldsymbol{x}) = L^*(s, \boldsymbol{y})\, (s, \boldsymbol{y}; t, \boldsymbol{x}). \tag{11.2}$$

Then, the second axiom extends it to the identity

$$-K(s, \boldsymbol{y}) f(\boldsymbol{y}) = L^*(s, \boldsymbol{y}) f(\boldsymbol{y}) \tag{12.2}$$

where f is an arbitrary function. Therefore, we obtain

$$-K(s, \boldsymbol{y}) = L^*(s, \boldsymbol{y}) \tag{13.2}$$

and

$$\begin{aligned} b_{kj}(s, \boldsymbol{y}) + b^*_{kj}(s, \boldsymbol{y}) &= 0 \\ a_k(s, \boldsymbol{y}) - a^*_k(s, \boldsymbol{y}) &= \partial_j b_{kj}(s, \boldsymbol{y}) \\ c(s, \boldsymbol{y}) + c^*(s, \boldsymbol{y}) &= \partial_k a_k(s, \boldsymbol{y}) - \frac{1}{2} \partial_k \partial_j b_{kj}(s, \boldsymbol{y}). \end{aligned} \tag{14.2}$$

One sees from this that

$$\begin{aligned} b_{kj} &= i\beta_{kj} \\ a_k &= \xi_k + \frac{1}{2} \partial_j b_{kj} \\ c &= \frac{1}{2} \partial_k \xi_k + i\eta \end{aligned} \tag{15.2}$$

where β_{kj}, ξ_k and η are arbitrary real functions.

It is not difficult to prove, using (13.2), that both operators K and L are antisel fadjoint on those functions from $L^2(\mathscr{X})$ which vanish together with their derivatives on the boundary of $\mathscr{X}$. Therefore, we get the most general Schrödinger equations for wave functions ψ and φ defined as

$$\psi(t, \boldsymbol{x}) = \int_{\mathscr{X}} d\boldsymbol{y} u(\boldsymbol{y}) \, (s, \boldsymbol{y}; t, \boldsymbol{x})$$

$$\varphi(s, \boldsymbol{y}) = \int_{\mathscr{X}} d\boldsymbol{x} \, (s, \boldsymbol{y}; t, \boldsymbol{x}) \, u^*(\boldsymbol{x}), \quad u \in L^2(\mathscr{X}), \tag{16.2}$$

$$\partial_t \psi(t, \boldsymbol{x}) = L(t, \boldsymbol{x}) \, \psi(t, \boldsymbol{x}) \tag{17.2}$$

$$-\partial_s \varphi(s, \boldsymbol{y}) = K(s, \boldsymbol{y}) \, \varphi(s, \boldsymbol{y}). \tag{18.2}$$

It is easy to check by using the unitarity condition (iii) that the norms of ψ and φ exist and are constant in time. The function $\psi(t, \boldsymbol{x})$ gives the probability amplitude of finding a particle at the point $\boldsymbol{x}$ in the future $t > s$ if the initial wave function is u. The second wave function $\varphi(s, \boldsymbol{y})$ gives the probabilistic description of the particle in the past. We have, according to the first axiom,

$$\psi(t, \boldsymbol{x}) = \varphi^*(t, \boldsymbol{x}) \tag{19.2}$$

$$\varphi(s, \boldsymbol{y}) = \psi^*(s, \boldsymbol{y}).$$

Using these equations it is easy to prove that

$$\partial_t |\psi|^2 + \operatorname{div} \boldsymbol{J}(\psi) = 0 \tag{20.2}$$

$$J_k(\psi) = \left[a_k - \frac{1}{2} (\partial_j b_{kj}) \right] |\psi|^2 + \frac{1}{2} b_{kj} (\psi \partial_j \psi^* - \psi^* \partial_j \psi).$$

One sees that only the real part of a_k enters this continuity equation. A similar equation holds for $|\varphi|^2$ and may be obtained from the last equation by making the substitution $\psi \to \varphi^*$.

3. *Electromagnetic structure of the Schrödinger equations*

Our next problem is to determine the functional dependence of the phenomenological coefficients a_k, b_{kj} and c, describing a charged system moving in an external electromagnetic field, on the potentials $\boldsymbol{A}$ and φ

$$\boldsymbol{E} = -\frac{1}{c} \dot{\boldsymbol{A}} - \nabla \varphi$$

$$\boldsymbol{H} = \operatorname{rot} \boldsymbol{A}. \tag{1.3}$$

The known prescription [18] for finding the Hamiltonian for such a problem is based on analogy with classical mechanics and for this reason cannot be used here. However, we shall argue that the substitution

$$\nabla \to \nabla + i\lambda \boldsymbol{A} \tag{2.3}$$

$$\partial_s \to \partial_s - i\lambda c \varphi$$

in the equation (9.2) and a similar substitution with reversed sign of λ in the equation (10.2) gives the most general explicite dependence of coefficients in the Schrödinger equations on electromagnetic potentials. In order to make this clear we demand, first, the physical equivalence of the equations

$$[\partial_s + K(s, \boldsymbol{y}; \boldsymbol{A}, \varphi)]\,(s, \boldsymbol{y}; t, \boldsymbol{x}) = 0 \tag{3.3}$$

and

$$\left[\partial_s + K\left(s, \boldsymbol{y}; \boldsymbol{A} + \nabla \Lambda, \varphi - \frac{1}{c}\dot{\Lambda}\right)\right](s, \boldsymbol{y}; t, \boldsymbol{x})' = 0 \tag{4.3}$$

where Λ is any gauge function. By physical equivalence we mean that both solutions $(s, \boldsymbol{y}; t, \boldsymbol{x})$ and $(s, \boldsymbol{y}; t, \boldsymbol{x})'$ are connected as follows:

$$(s, \boldsymbol{y}; t, \boldsymbol{x})' = (s, \boldsymbol{y}; t, \boldsymbol{x}) \exp\{iF(s, \boldsymbol{y}; \Lambda)\} \tag{5.3}$$

where F is some real function depending functionally on Λ. This dependence must be linear because the gauge transformations form an Abelian group. Moreover, F should be a homogeneous function of Λ, since both equations coincide at $\Lambda = 0$. Therefore, we have

$$F(s, \boldsymbol{y}; \Lambda) = \int dr \int d\boldsymbol{z}\, F(s, \boldsymbol{y}; r, \boldsymbol{z})\, \Lambda(r, \boldsymbol{z}). \tag{6.3}$$

The substitution of $(s, \boldsymbol{y}; t, \boldsymbol{x})$ evaluated from (5.3) into the equation (3.3) and a comparison of coefficients gives us the following functional equations

$$b_{kj}\left[\boldsymbol{A} + \nabla \Lambda, \varphi - \frac{1}{c}\dot{\Lambda}\right] = b_{kj}[\boldsymbol{A}, \varphi] \tag{7.3}$$

$$a_k\left[\boldsymbol{A} + \nabla \Lambda, \varphi - \frac{1}{c}\dot{\Lambda}\right] = a_k[\boldsymbol{A}, \varphi] - b_{kj}[\boldsymbol{A}, \varphi] F'_j \tag{8.3}$$

$$c\left[\boldsymbol{A} + \nabla \Lambda, \varphi - \frac{1}{c}\dot{\Lambda}\right] = c[\boldsymbol{A}, \varphi] - i\dot{F} - ia_k[\boldsymbol{A}, \varphi] F'_k -$$

$$- \frac{1}{2} b_{kj}[\boldsymbol{A}, \varphi] F'_k F'_j - \frac{i}{2} b_{kj}[\boldsymbol{A}, \varphi] F''_{kj}. \tag{9.3}$$

We shall solve them in two steps. First we shall determine what is called "the minimal electromagnetic coupling" solution, assuming that the functionals a_k, b_{kj} and c do not depend on $\boldsymbol{E}$ and $\boldsymbol{H}$. In fact, the above equations give us only an explicite dependence of functionals on the potentials $\boldsymbol{A}$ and φ. Later, we shall generalize this minimal solution to one which accounts for a possible dependence on $\boldsymbol{E}$ and $\boldsymbol{H}$.

It is clear from the first equation that b_{kj} does not depend explicitely on the potentials. The second equation shows then that a_k does not depend on φ because its increment does not depend on $\dot{\Lambda}$. Furthermore, in order to make the increment of a_k, which is $-ib_{kj}[0]\,F'_j$, dependent on Λ'_k instead of Λ itself, we must impose the locality requirement

$$F(s, \boldsymbol{y}; r, \boldsymbol{z}) = \lambda(s)\, \delta(s - r)\, \delta(\boldsymbol{y} - \boldsymbol{z}) \tag{10.3}$$

$$F(s, \boldsymbol{y}; \Lambda) = \lambda \cdot \Lambda(s, \boldsymbol{y})$$

where λ is a real function of s. In this case we obtain from (8.3) the equation

$$a_k[\boldsymbol{A}+\nabla\Lambda] = a_k[\boldsymbol{A}]-i\lambda b_{kj}[0]\,\Lambda'_j. \qquad (11.3)$$

Putting $\boldsymbol{A}=0$ and redenoting again $\nabla\Lambda$ as $\boldsymbol{A}$, we get

$$a_k[\boldsymbol{A}] = a_k[0]-i\lambda b_{kj}[0]\,A_j. \qquad (12.3)$$

In order to solve the equation (9.3) we notice first that the increment of c does not depend on the products $\dot{\Lambda}\Lambda'_k$, what means that $c[\boldsymbol{A},\varphi]$ has the structure

$$c[\boldsymbol{A},\varphi] = \eta[\varphi]+\zeta[\boldsymbol{A}] \qquad (13.3)$$

where the functionals η and ζ satisfy the equations

$$\eta\left[\varphi-\frac{1}{c}\dot{\Lambda}\right] = \eta[\varphi]-i\dot{\lambda}\Lambda-i\lambda\dot{\Lambda} \qquad (14.3)$$

$$\zeta[\boldsymbol{A}+\nabla\Lambda] = \zeta[\boldsymbol{A}]-i\lambda(a_k[0]-i\lambda b_{kj}[0]A_j)\Lambda'_k-$$

$$-\frac{\lambda^2}{2}b_{kj}[0]\Lambda'_k\Lambda'_j-\frac{i}{2}\lambda b_{kj}[0]\Lambda''_{kj}. \qquad (15.3)$$

From these equations we conclude that

$$\lambda = \text{const}$$

$$\eta[\varphi] = \eta[0]+ic\lambda\varphi \qquad (16.3)$$

and

$$\zeta[\boldsymbol{A}] = \zeta[0]-i\lambda a_k[0]A_k-\frac{\lambda^2}{2}b_{kj}[0]A_kA_j-\frac{i\lambda}{2}b_{kj}[0]\partial_kA_j.$$

Collecting the results we have the following minimal solution:

$$b_{kj}[\boldsymbol{A},\varphi] = b_{kj}[0]$$

$$a_k[\boldsymbol{A},\varphi] = a_k[0]-i\lambda b_{kj}[0]A_j$$

$$c[\boldsymbol{A},\varphi] = c[0]+ic\lambda\varphi-i\lambda a_k[0]A_k-\frac{\lambda^2}{2}b_{kj}[0]A_kA_j-\frac{i\lambda}{2}b_{kj}[0]\partial_kA_j. \qquad (17.3)$$

In order to get the most general solution of the equations (7.3), (8.3) and (9.3) one should admit that $a_k[0]$, $b_{kj}[0]$ and $c[0]$ are arbitrary functionals or $\boldsymbol{E}$ and $\boldsymbol{H}$. There is only one general constraint following from the formulae (15.2). Namely, because of the relations

$$\partial_k\,\text{Re}\,a_k[\boldsymbol{A},\varphi] = 2\,\text{Re}\,C[\boldsymbol{A},\varphi]$$

$$\text{Im}\,a_k[\boldsymbol{A},\varphi] = -\frac{i}{2}\partial_jb_{kj}[\boldsymbol{A},\varphi] \qquad (18.3)$$

we get from (17.3) the condition

$$\partial_k\text{Re}\,a_k[0] = 2\,\text{Re}\,c[0]. \qquad (19.3)$$

In order to distinguish the minimal and the general solutions we shall use the following notation in the latter case

$$a_k[0] = a_k[\boldsymbol{E}, \boldsymbol{H}] \tag{20.3}$$

and likewise for $b_{kj}[0]$ and $c[0]$.

In this way we arrived at the Schrödinger equations for a particle moving in an external electromagnetic field,

$$\{\partial_s - ic\lambda\varphi + \tfrac{1}{2}b_{kj}[\boldsymbol{E}, \boldsymbol{H}]\,(\partial_k - i\lambda A_k)\,(\partial_j - i\lambda A_j) + \\ + a_k[\boldsymbol{E}, \boldsymbol{H}]\,(\partial_k - i\lambda A_k) + c[\boldsymbol{E}, \boldsymbol{H}]\}\,\varphi(s, \boldsymbol{y}) = 0 \tag{21.3}$$

$$\{-\partial_t + ic\lambda\varphi + \tfrac{1}{2}\,(\partial_k + i\lambda A_k)\,(\partial_j + i\lambda A_j)\,b_{kj}[\boldsymbol{E}, \boldsymbol{H}] - \\ -(\partial_k + i\lambda A_k)\,a_k[\boldsymbol{E}, \boldsymbol{H}] + c[\boldsymbol{E}, \boldsymbol{H}]\}\psi(t, \boldsymbol{x}) = 0. \tag{22.3}$$

The standard dependence of λ on a charge q of a particle is

$$q = -\lambda\hbar c. \tag{23.3}$$

From the corresponding equations for the amplitude we conclude that

$$(s, \boldsymbol{y}; t, \boldsymbol{x})_\gamma = (s, \boldsymbol{y}; t, \boldsymbol{x})_0 \exp i\lambda\int_\gamma \varphi c d\tau - \boldsymbol{A}d\boldsymbol{z} \tag{24.3}$$

where $(s, \boldsymbol{y}; t, \boldsymbol{x})_0$ solves the equation

$$\{\partial_s + \tfrac{1}{2}b_{kj}[\boldsymbol{E}, \boldsymbol{H}]\,\partial_k\partial_j + a_k[\boldsymbol{E}, \boldsymbol{H}]\,\partial_k + c[\boldsymbol{E}, \boldsymbol{H}]\}\,(s, \boldsymbol{y}; t, \boldsymbol{x}) = 0 \tag{25.3}$$

and the integration in the exponent is carried over some path γ connecting $(s, \boldsymbol{y})$ and $(t, \boldsymbol{x})$. There is no path dependence of the amplitude only if the external electromagnetic field is absent.

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