UNITARITY OF THE COLLISION MATRIX AND INTERDEPENDENCE OF RESONANCE PARAMETERS

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The cross-sections for the reactions between the various channels of a compound nuclear system within a given energy interval are determined by contributions from a finite number of intrinsic resonant states of the system, superposed upon a smooth background. The unitarity and symmetry of the collision matrix impose a set of conditions on the parameters characterizing the resonant states and the background. These conditions are written explicitly in general form for the simple case of an energy interval far from the thresholds of all the open channels.

1. Introduction

Experiments on reactions between the various channels of a compound nuclear system within a given energy interval often show that the cross-section variation with energy is dominated by a finite number of resonant states of the compound system, but interfering among themselves and with the smooth background upon which they are superposed. In order to extract from such results the values of the parameters characterizing the resonances, a rational method is provided by the representation of the collision matrix in terms of its complex poles in the physical energy plane. However, in attempting to adjust the parameters entering such a representation so as to fit the experimental data, one must not lose sight of the fact that these parameters are not independent of each other, but have to satisfy a set of conditions resulting from the unitarity and symmetry of the collision matrix. If, therefore, a fit of the data has been obtained by independent adjustment of the parameters, it is still necessary — especially if the effect of interferences is such as to raise doubts about the unambiguity of the fit — to check whether the parameters thus found satisfy the unitarity and symmetry conditions with sufficient accuracy. To this end, it is desirable to have at one's disposal an explicit specification of the unitarity and symmetry constraints imposed upon the parameters. Although the statement of such constraints raises no difficulty of principle, their explicit formulation under the most general conditions is quite a compli-

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cated task, owing to the occurrence of the energy-dependent penetration factors of the reaction channels, which considerably distort the simple resonance structure of the collision matrix in the neighbourhood of the channel thresholds.

In order to appreciate this point, let us consider a typical element $\mathcal{U}_{c'c}(E)$ of the collision matrix pertaining to the reaction of a compound nuclear system between two channels c, c' at a given energy E. In terms of the intrinsic resonance states of the compound system, this matrix element has the form [1]

$$\mathcal{U}_{c'c}(E) = e^{i\varphi_{c'}(E)} \left\{ \mathcal{B}_{c'c}(E) - i \sum_n q_n \frac{G_{c'n}(E) G_{nc}(E)}{E - \mathcal{E}_n} \right\} e^{i\varphi_c(E)}; \tag{1}$$

it consists essentially of a series of resonance terms, corresponding to complex poles $\mathcal{E}_n$ of the collision matrix, and a background contribution represented by a smooth function of the energy $\mathcal{B}_{c'c}(E)$; the real constants q_n affecting the resonance terms are correction factors which are very near unity for narrow resonances. The residues of the poles $\mathcal{E}_n$ are factorized with respect to the channels; each factor $G_{cn}(E) \equiv G_{nc}(E)$ essentially represents, in absolute value, the square root of the partial width Γ_{cn} of the resonant state $\mathcal{E}_n$ in channel c, but it also contains, besides a constant phase factor exp $(i\zeta_{cn})$, an energy-dependent correction taking account of the limited penetrability of the channel and of the relative velocity of the fragments:

$$G_{cn}(E) = \sqrt{\frac{k_c(E) P_c(E)}{\operatorname{Re} k_c(\mathcal{E}_n) |P_c(\mathcal{E}_n)|}}\, g_{cn}, \qquad g_{cn} \equiv \Gamma_{cn}^{1/2} e^{i\zeta_{cn}}; \tag{2}$$

in this formula, $k_c(E)$ denotes the wave number in channel c and $P_c(E)$ the absolute value of the penetration factor in this channel, defined in such a way that it tends to unity for $E \to \infty$; to the penetration factor belongs also a phase factor exp $\{2i\varphi_c(E)\}$ which, however, as indicated in Eq. (1), is of no influence on the unitarity of the $\mathcal{U}$-matrix and will play no part in our discussion.

In the neighbourhood of the channel threshold, the penetration factor $P_c(E)$ may be given an expression completely independent of any channel radius and exhibiting the correct dependence of the channel wave number [1]; for this case, J. Humblet has recently obtained a parametrization of the complete collision matrix that is explicitly unitary [2]. This parametrization is not based directly, like Eq. (1), on the resonance poles, but on the poles of an auxiliary "$\mathcal{K}$-matrix" related to the collision matrix; the parameters of the resonance poles must accordingly be derived from those fitting the experimental data by an additional computation. The increased labour thus implied by such an approach appears, however, to be inherent in the particular physical situation envisaged. Unfortunately, the expression for the penetration factor adapted to the immediate neighbourhood of the threshold does not tend to unity for large values of the wave number and is therefore not suited to represent the physical penetration effect at higher energies. It would be possible to remedy this defect at the price of introducing an arbitrary channel radius [1], which would have to be treated as another adjustable parameter; but it is not clear at the moment whether Humblet's method can be applied to this more general case and thus provide a parametrization covering the whole energy domain.

On the other hand, if we renounce such a universal parametrization and restrict the scope of the problem to a determination of the collision matrix valid over a given energy interval, we may immediately single out a case amenable to a simpler treatment, namely that of an energy interval sufficiently far from every channel threshold. If we only retain in the form (1) of the collision matrix those resonance terms whose effect is prominent in the energy interval considered, and include the others into the background contribution, we may regard the whole energy-dependent factor in the expression (2) for the $G_{cn}(E)$ as constant (and nearly unity) over the energy interval; at any rate, Eq. (2) allows us to estimate in each concrete case the error involved in this simplification. In many cases, it will further be permissible to neglect the energy-dependence of the background amplitude over the energy interval, and accordingly reduce the problem to the consideration of a unitary matrix of the type

$$U_{c'c} = B_{c'c} - i \sum_n q_n \frac{g_{c'n}\, g_{nc}}{E - \mathscr{E}_n}, \tag{3}$$

involving only a finite sum and constant parameters. This limiting case, in spite of its severe limitations, is well worth investigating as a counterpart to that treated by Humblet; the comparison of the two opposite extremes might suggest a way of bridging the gap between them. The method to deal with matrices of the type (3) has been outlined by McVoy [3], but since he only applied it to cases not involving more than two resonances, the general form of his results is not immediately apparent. The aim of this note will be to give a more systematic treatment of McVoy's method, applicable to any number of resonances and channels.

2. *The K-matrix*

Let us first recall that a matrix of the type (3), with constant parameters, can be further reduced by elimination of the background parameters. This reduction, pointed out by McVoy [3, 4], is based on the fact that the unitarity of the matrix U implies that of the matrix B: for, if we reduce the terms of the expression (3) to the same denominator, the matrix B appears at the coefficient of the highest power of E in the numerator. The eigenvalues of B may therefore be written in the form $\exp\,(2i\beta_k)$; since B must also be symmetrical, the corresponding eigenvectors χ_{ck} may be taken to be real, and if they are normalized to unity, they form an orthogonal matrix. The matrix b defined by

$$b_{ck} = e^{i\beta_k} \chi_{ck} \tag{4}$$

is then unitary, and it yields for B the expression $B = b\tilde{b}$. We have now only to define the transforms $u_{kn} \equiv u_{nk}$ of the partial width parameters by the inverse matrix b^{-1} or b^{+}:

$$u_{kn} = u_{nk} = q_n^{1/2} \sum_{c'} g_{nc'} b^*_{c'k} \tag{5}$$

in order to bring the matrix U into the form

$$U = bS\tilde{b},$$

$$S_{k'k} = \delta_{k'k} - i \sum_n \frac{u_{k'n} u_{nk}}{E - \mathscr{E}_n}; \tag{6}$$

the matrix S is again a symmetrical and unitary matrix with the same resonance poles as U, but all its constant parameters now refer to the resonances. If we restrict ourselves to N resonances and C open channels, the number of real constant parameters occurring in S is $2N(C+1)$. Only half this number is freely adjustable, however, since the unitarity and symmetry of S leads to $N(C+1)$ relations between them[1].

Now, a natural approach to the unitarity relations of the matrix S is offered by the representation of S with the help of a hermitian matrix K:

$$S = \frac{\delta + iK}{\delta - iK}, \tag{7}$$

where δ denotes the unit matrix of the appropriate rank. Since the matrix S is symmetrical, K has the same property and all its elements are therefore real. The unitarity and symmetry conditions for S are thus equivalent to the reality conditions for K, and the latter are easier to handle. The decisive point in this connection is the fact that the elements of the matrix K, like those of S, are meromorphic functions of the energy, and that the properties of the matrix K entail the reality of its poles and residues. Since these parameters can be expressed in terms of the resonance parameters of the matrix S, the conditions resulting from their reality yield the required constraints upon the resonance parameters. This is the framework of McVoy's argument, which clearly indicates the steps we have now to take.

The first step is to establish the relations connecting the parameters which characterize the poles of the matrices S and K. By some easy algebra, the detail of which is given in the Appendix, one finds that the matrix K can be written in the form

$$K_{kk'} = -\tfrac{1}{2} \sum_\mu \frac{v_{k\mu}\, v_{\mu k'}}{E - \varepsilon_\mu}, \tag{8}$$

exhibiting explicitly its dependence on the poles ε_μ and a set of parameters $v_{k\mu} \equiv v_{\mu k}$, analogous to the partial width parameters $u_{kn} \equiv u_{nk}$ occurring in the expression (6) for the S-matrix; all the parameters ε_μ and $v_{k\mu}$ are real quantities. Moreover, they are related with the resonance parameters $\mathscr{E}_n$, u_{kn} by a complex-orthogonal transformation $O = ||O_{n\mu}||$ with $\tilde{O}O = \delta$; in matrix notation, these fundamental relations may be written

$$\mathscr{E} + \tfrac{1}{2} i \sum_k \tilde{u}^k u^k = O \varepsilon \tilde{O}, \tag{9}$$

$$u^k = v^k \tilde{O}, \qquad \tilde{u}^k = O \tilde{v}^k; \tag{10}$$

[1] There are C relations $\Sigma_{k'}|S_{k'k}|^2 = 1$ between the moduli of the matrix elements of S, and $C(C-1)$ further real relations $\Sigma_{k''}|S_{k'k''}||S_{k''k}| \exp[i(\varphi_{k'k''} - \varphi_{k''k})] = 0$ $(k' \neq k)$ between the $\frac{1}{2}C(C-1)$ non-arbitrary phases $\varphi_{k'k}$ and $\frac{1}{2}C(C-1)-1$ ratios of independent moduli, leaving after elimination only one more independent relation between these quantities. Each of the $(C+1)$ relations thus enumerated may be put into the form of an identity of the N-th degree in E, with a fixed coefficient of E^N: we therefore obtain altogether $N(C+1)$ conditions for the resonance parameters.

in these formulae, u^k (and similarly v^k) represents the one-row matrix $||u_{k1} \dots u_{kN}||$, and $\tilde{u}^k$ the transposed one-column matrix with the same elements; $\mathscr{E}$ and ε denote the diagonal matrices $||\mathscr{E}_n \delta_{nn'}||$, $||\varepsilon_\mu \delta_{\mu\mu'}||$.

The transformation O, as shown in the Appendix, may be put into the form

$$O = \gamma R \tag{11}$$

of the product of a real matrix R of non-vanishing determinant and a hermitian matrix of the form

$$\gamma = \delta + i\beta, \tag{12}$$

where β is a real and antisymmetrical matrix. The relations (9) and (10), together with the decomposition (11), (12) of the O-transformation, are all we need to derive the constraints on the resonance parameters.

3. The reality conditions for the K-matrix

We must now formulate, with the help of the relations (9), (10), the conditions which express the reality of the N poles ε_μ and the NC constants of the K-matrix.

In the first instance, these conditions will thus involve, besides the resonance parameters, those entering into the expression of the transformation matrix. As a final step, the latter parameters will themselves be expressed in terms of the resonance parameters. In practical applications, it will be appropriate to follow these two steps in the reverse order.

Beginning with the partial width parameters u^k, let us separate them into their real and imaginary parts:

$$u^k = c^k + i s^k. \tag{13}$$

On account of Eqs (10) and (11), the reality of v^k leads at once to the NC constraints

$$s^k = -c^k \beta, \tag{14}$$

which may be regarded as limitations affecting the phases of the partial width parameters in the various channels. In fact, inverting the relations (5) and taking account of Eqs (4) and (2), we have, for each channel separately in virtue of the conditions (14),

$$(q_n \Gamma_{cn})^{1/2} e^{i\zeta_{cn}} = \sum_{k,n'} \chi_{ck} c_{kn'} e^{i\beta_k} (\delta_{n'n} - i\beta_{n'n}); \tag{15}$$

from Eqs (15) one readily finds

$$\operatorname{tg} \zeta_{cn} = \frac{\sigma_{cn} + \sum_{n'} \varkappa_{cn'} \beta_{n'n}}{\varkappa_{cn} - \sum_{n'} \sigma_{cn'} \beta_{n'n}}, \tag{16}$$

where the quantities

$$\sigma_{cn} = \sum_k \chi_{ck} c_{kn} \sin \beta_k, \qquad \varkappa_{cn} = \sum_k \chi_{ck} c_{kn} \cos \beta_k \tag{17}$$

only depend on the real parts of the partial width parameters u_{kn} and the background parameters.

Much simpler relations are obtained if the background matrix is diagonal, $B_{c'c} = \delta_{c'c} \exp(i\beta_c)$. Eqs (15) then reduce to

$$(q_n \Gamma_{cn})^{1/2} e^{i(\zeta_{cn}-\beta_c)} = \sum_{n'} c_{cn'}(\delta_{n'n} - i\beta_{n'n}), \tag{15a}$$

whence

$$(q_n \Gamma_{cn})^{1/2} \sin(\zeta_{cn}-\beta_c) = \sum_{n'} \beta_{nn'}(q_{n'}\Gamma_{cn'})^{1/2} \cos(\zeta_{cn'}-\beta_c) \tag{16a}$$

for every channel. Another relation readily derived from Eqs (15*a*) on account of the anti-symmetry of β is the following:

$$\sum_n (q_n \Gamma_{cn})^{1/2} (q_n \Gamma_{c'n})^{1/2} \sin(\zeta_{cn}-\beta_c+\zeta_{cn}-\beta_{c'}) = 0, \tag{17a}$$

valid for every pair of channels; this type of condition has been pointed out by Lynn (Ref. [6] p. 87–88). In particular, for every channel,

$$\sum_n q_n \Gamma_{cn} \sin 2(\zeta_{cn}-\beta_c) = 0, \tag{18a}$$

a relation which is also an immediate consequence of Eqs (16a).

In order to obtain the N remaining constraints, we must now turn to the relations (9). On account of Eqs (11), the right-hand side may be written $\gamma\eta\tilde{\gamma}$, where $\eta = R\varepsilon\tilde{R}$ is a real and symmetrical matrix. We must make use of this fact, but in such a way as to eliminate the matrix η, which depends on the unknown poles ε_μ of the K-matrix. A convenient procedure is to transform the matrix $\gamma\eta\tilde{\gamma}$ into a hermitian matrix, which is achieved by multiplying it on the right with the matrix $\theta = \tilde{\gamma}^{-1}\gamma$. After performing the same operation on the left-hand side of Eq. (9), we obtain the desired conditions by stating that the anti--hermitian part of the resulting expression must vanish. Now, on account of the conditions (14) and the hermitian character of the matrix γ, we have

$$u^k = c^k\tilde{\gamma}, \qquad u^{k*} = c^k\gamma;$$

the effect of the multiplication of the expression $\sum_k \tilde{u}^k u^k$ with the matrix θ is therefore to transform it into the hermitian matrix

$$G = \sum_k \tilde{u}^k u^{k*}, \tag{18}$$

and the left-hand side of Eq. (9) becomes $\mathscr{E}\theta + \frac{1}{2}iG$. Since the matrix θ is hermitian, the required conditions take the form

$$G = i(\mathscr{E}\theta - \theta\mathscr{E}^*), \tag{19}$$

or more explicitly

$$G_{nn'} = i\theta_{nn'}(\mathscr{E}_n - \mathscr{E}^*_{n'}). \tag{20}$$

These conditions depend only on the matrix θ and on the resonance parameters; in fact, we have, in virtue of Eqs (2) and (5)

$$G_{nn'} = \sum_c (q_n \Gamma_{nc})^{1/2} e^{i(\zeta_{nc}-\zeta_{cn'})} (q_{n'}\Gamma_{cn'})^{1/2}. \tag{21}$$

Equating the two expressions for the diagonal elements G_{nn} given by Eqs (20) and (21) yields the relation

$$q_n \sum_c \Gamma_{nc} = \theta_{nn} \Gamma_n;$$

this may be interpreted as the statement that the total width Γ_n of a resonance is the sum of its partial widths in the available channels[2],

$$\Gamma_n = \sum_c \Gamma_{nc}, \tag{22}$$

provided that we identify the correction factor q_n occurring in the corresponding resonance term of the collision matrix (3) with the quantity θ_{nn}; in other words, the conditions $q_n = \theta_{nn}$ are equivalent to the relations (22) between the resonance parameters, and may therefore be taken as the formulation of the N last constraints we require. They clearly show the origin of the factors q_n — which we already know from the structural analysis of the collision matrix [1] — in the passage from symmetrical matrices (demanded by time-reversal invariance) to hermitian matrices (demanded by unitarity). The matrix θ, which effects this passage, is readily expressed in terms of the elements of the matrix β, by means of the minors $C_{nn'}$ of the matrix $\tilde{\gamma}$ and its real determinant $C = \det \gamma$. In fact, if $C_{nn'}$ is defined as the minor of the element $\tilde{\gamma}_{n'n}$, we have

$$\theta_{nn'} = C^{-1} \sum_{n''} C_{nn''}(\delta_{n''n'} + i\beta_{n''n'}) = C^{-1}[C_{nn'} + i(C\beta)_{nn'}];$$

by definition, however,

$$C\delta_{nn'} = \sum_{n''} C_{nn''}(\delta_{n''n'} - i\beta_{n''n'}) = C_{nn'} - i(C\beta)_{nn'}. \tag{23}$$

Hence,

$$\theta_{nn'} = 2C^{-1}C_{nn'} - \delta_{nn'}. \tag{24}$$

The conditions on the factors q_n accordingly take the form

$$q_n = \frac{2\,C_{nn} - C}{C}, \tag{25}$$

where the determinants C, C_{nn} can easily be written down, in each concrete case, in terms of the matrix elements $\beta_{nn'}$.

All the constraints we have found, given by Eqs (14) and (25), depend only on the antisymmmetrical matrix β. Our last task is therefore to express this matrix in terms of the resonance parameters. For this purpose, we need $\frac{1}{2}N(N-1)$ real equations; a simple set of such equations may be derived from the equations (20) for $n \neq n'$, which we have not yet used. Combining these with Eqs (21) and (24), we get

$$\frac{2\,C_{nn'}}{C(q_n\, q_{n'})^{\frac{1}{2}}} = -\frac{i}{|\mathscr{E}_n - \mathscr{E}_{n'}{}^*|^2} \sum_c (\Gamma_{nc}\Gamma_{cn'})^{\frac{1}{2}} (\mathscr{E}_n^* - \mathscr{E}_{n'}) e^{i(\zeta_{nc} - \zeta_{cn'})}; \tag{26}$$

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[2] In contrast to the total resonance width Γ_n, the partial widths Γ_{nc} associated with the resonance vary according to the number of open channels.

Eq. (23), which may be written out, for $n \neq n'$, as

$$C_{nn'} = iC_{nn}\beta_{nn'} + i \sum_{n''\neq n} C_{nn''}\beta_{n''n'}, \tag{27}$$

suggests that a simpler set of equations will result from the imaginary part of Eqs (26) than from their real part:

$$\frac{2\,\mathrm{Im}\,C_{nn'}}{C(q_n q_{n'})^{\frac{1}{2}}} = \frac{1}{|\mathscr{E}_n - \mathscr{E}^*_{n'}|^2} \sum_c (\Gamma_{nc}\Gamma_{cn'})^{\frac{1}{2}} [-(E_n - E_{n'}) \cos(\zeta_{nc} - \zeta_{cn'}) + \tag{28}$$

$$+\tfrac{1}{2}(\Gamma_n + \Gamma_{n'}) \sin(\zeta_{nc} - \zeta_{cn'})].$$

In using these equations for the determination of the $\beta_{nn'}$, it may be advantageous to replace the quantities q_n, $q_{n'}$ by their expressions (25), and perhaps also to take into account the other constraints (14) on the phases ζ_{nc}. Eqs (27) provide the basis for a convenient iteration procedure for finding the explicit expression of $C_{nn'}$ as a rational function of the $\beta_{nn'}$; the number of iterations is limited by the degree, $N-1$, of this rational function.

In usual situations, we shall not expect the transformation matrix γ to differ much from the unit matrix, and we are accordingly interested in possible solutions of the Eqs (28) for which the values of the $(\beta_{nn'})^2$ will not be large; as pointed out at the end of the Appendix, the inequality $|C| < 1$, which is a general property of the transformation γ, is satisfied in the domain of values $(\beta_{nn'})^2 < 1$, in which the dominant terms of the determinants C and $C_{nn'}$ take very simple forms:

$$\begin{aligned} C &= 1 - \sum_{n_1 < n_2} (\beta_{n_1 n_2})^2, \\ C_{nn} &= C + \sum_{n'} (\beta_{nn'})^2, \qquad\qquad [N = 2 \text{ or } 3] \\ C_{nn'} &= i\beta_{nn'} - \sum_{n''} \beta_{nn''}\beta_{n''n'}, \qquad (n \neq n'); \end{aligned} \tag{29}$$

as indicated, these formulae are exact in the cases of two or three resonances. The expression for C_{nn} is obtained from that of C by putting $\beta_{nn'} = 0$ for the fixed value of n and all n'. Accordingly, the expression (25) for q_n may be written as C^*_n/C, where C^*_n denotes the determinant C in which the sign of the terms containing the particular $\beta_{nn'}$ just specified has been reversed; limited to the dominant terms, this gives

$$Cq_n = C^*_n = 1 - \sum_{n_1 < n_2} (\beta_{n_1 n_2})^2 + 2 \sum_{n'} (\beta_{nn'})^2, \qquad [N = 2 \text{ or } 3] \tag{30}$$

a formula exact for $N = 2$ or 3.

Under the assumption $(\beta_{nn'})^2 < 1$, we arrive at a narrower estimate of the upper bound of the $(\beta_{nn'})^2$ by starting from the Schwarz inequality which, in view of the definition (18) of the matrix G, holds for its elements in the form

$$|G_{nn'}|^2 \leqslant G_{nn}G_{n'n'}.$$

Inserting in this inequality the expression (20) for $G_{nn'}$ and the values $G_{nn}=q_n\Gamma_n$, $G_{n'n'}=q_{n'}\Gamma_{n'}$ we get

$$\frac{|\theta_{nn'}|^2}{q_n q_{n'}} \leqslant \frac{\Gamma_n \Gamma_{n'}}{(E_n-E_{n'})^2+\frac{1}{4}(\Gamma_n+\Gamma_{n'})^2}.$$

The maximum value of the right-hand side is unity; it is not attained in our case, since it would correspond to both $E_n = E_{n'}$, and $\Gamma_n = \Gamma_{n'}$, *i.e.* two coinciding resonance poles, an eventuality obviously excluded from our analysis. On account of Eqs (24), we have therefore, for $n \neq n'$,

$$\frac{|2C_{nn'}|^2}{C^2 q_n q_{n'}} \leqslant \frac{\Gamma_n \Gamma_{n'}}{(E_n-E_{n'})^2+\frac{1}{4}(\Gamma_n+\Gamma_{n'})^2} < 1. \tag{31}$$

On the left-hand side, the denominator may obviously be replaced by the square of the arithmetical means $\frac{1}{2}(Cq_n+Cq_{n'})$, for which Eqs (30) readily yield the inequality

$$\tfrac{1}{2}C(q_n+q_{n'}) \leqslant 1+(\beta_{nn'})^2.$$

Moreover, under neglect of terms of the fourth and higher degrees in the $\beta_{nn'}$, we may write, according to the last Eqs (29),

$$(\beta_{nn'})^2 \lesssim |C_{nn'}|^2.$$

We therefore obtain from the inequality (31) the following approximate one for the matrix element $\beta_{nn'}$:

$$\frac{(2\beta_{nn'})^2}{[1+(\beta_{nn'})^2]^2} \lesssim \frac{\Gamma_n \Gamma_{n'}}{(E_n-E_{n'})^2+\frac{1}{4}(\Gamma_n+\Gamma_{n'})^2} < 1. \tag{32}$$

Denoting by $(\xi_{nn'})^2$ the ratio of the smaller to the larger of the widths Γ_n, $\Gamma_{n'}$ we derive from the inequality (32) the further one

$$\frac{(2\beta_{nn'})^2}{[1+(\beta_{nn'})^2]^2} \lesssim \frac{(2\xi_{nn'})^2}{[1+(\xi_{nn'})^2]^2}.$$

In conjunction with our assumption $(\beta_{nn'})^2 < 1$, this inequality finally yields

$$(\beta_{nn'})^2 \lesssim (\xi_{nn'})^2. \tag{33}$$

It remains to illustrate the preceding general discussion by its application to the simplest cases of two or three resonances.

3. *Applications*

For the case of 3 resonances, we shall merely write out the equations (25) and (28), since the phase constraints (14) do not present any special features. The matrix β will be written in the simpler notation

$$\beta \equiv \begin{Vmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{Vmatrix}.$$

Eqs (29) have the exact form

$$C = 1-\beta_1^2-\beta_2^2-\beta_3^2; \quad C_{11} = 1-\beta_1^2, \ldots; \quad C_{12} = i\beta_3-\beta_1\beta_2, \ldots;$$

hence, Eqs (30) become

$$q_1 = \frac{1-\beta_1^2+\beta_2^2+\beta_3^2}{1-\beta_1^2-\beta_2^2-\beta_3^2}, \ldots$$

and Eqs (28) are of the type

$$\frac{2\beta_3}{\sqrt{(1+\beta_3^2)^2-(\beta_1^2-\beta_2^2)^2}} = \sum_c \frac{(\Gamma_{1c}\Gamma_{2c})^{\frac{1}{2}}}{(E_1-E_2)^2+\frac{1}{4}(\Gamma_1+\Gamma_2)^2}[-(E_1-E_2)\cos(\zeta_{1c}-\zeta_{2c})+$$
$$+\tfrac{1}{2}(\Gamma_1+\Gamma_2)\sin(\zeta_{1c}-\zeta_{2c})], \ldots$$

We shall not develop the algebra of this case further, but rather turn to the case of two resonances, in which the calculations become quite transparent.

For $N = 2$, the β-matrix reduces to a single parameter, which we shall denote as β since no confusion has to be feared: the β-matrix is just

$$\begin{Vmatrix} 0 & \beta \\ -\beta & 0 \end{Vmatrix}.$$

Now, we have

$$C = 1-\beta^2, \quad C_{11} = C_{22} = 1, \quad C_{12} = i\beta$$

and we derive the important consequence that the two correction factors q_1, q_2 must be equal and larger than unity:

$$q_1 = q_2 \equiv q = \frac{1+\beta^2}{1-\beta^2}. \tag{34}$$

The equation determining β may be written

$$\frac{2\beta}{1+\beta^2} = \sum_c \frac{(\Gamma_{1c}\Gamma_{2c})^{\frac{1}{2}}}{(E_1-E_2)^2+\frac{1}{4}(\Gamma_1+\Gamma_2)^2}[-(E_1-E_2)\cos(\zeta_{1c}-\zeta_{2c})+\tfrac{1}{2}(\Gamma_1+\Gamma_2)\sin(\zeta_{1c}-\zeta_{2c})]. \tag{35}$$

In this case[3], the inequality (31) takes immediately the form (32), leading to the upper bound $\beta^2 \leqslant \xi^2$, where ξ^2 is the ratio of the smaller to the larger of the widths Γ_1, Γ_2.

[3] The possible use of Schwarz' inequality was noticed by McVoy [3]. However, this inequality does not lead, as he asserts, to the conclusion that β^2 must be < 1. Only the assumption $\beta^2 < 1$ allows us to derive from the inequality (32) the upper bound $\beta^2 \leqslant \xi^2 < 1$.

Let us now turn to the phase constraints, under the assumption of a diagonal background matrix. With the notation $Z_{nc} = \zeta_{nc} - \beta_c$, Eqs (16$a$) become, because of Eq. (34),

$$\sqrt{\Gamma_{1c}} \sin Z_{1c} = \beta \sqrt{\Gamma_{2c}} \cos Z_{2c},$$

$$\sqrt{\Gamma_{2c}} \sin Z_{2c} = -\beta \sqrt{\Gamma_{1c}} \cos Z_{1c}. \tag{36}$$

An equivalent form for this set is the following:

$$\Gamma_{1c} \sin 2Z_{1c} + \Gamma_{2c} \sin 2Z_{2c} = 0, \tag{37}$$

$$\operatorname{tg} Z_{1c} \operatorname{tg} Z_{2c} = -\beta^2. \tag{38}$$

Eq. (36) is the special form, for $N = 2$, of the general equation (18a); Eq. (37), however, is peculiar to the case of 2 resonances and provides a strong restriction on the phases Z_{1c}, Z_{2c} in each channel, independently of the channel widths. By means of Eqs (36) and (38), we readily find

$$\cos(\zeta_{1c} - \zeta_{2c}) = \xi_c \frac{1-\beta^2}{2\beta} \sin 2\, Z_{1c},$$

$$\sin(\zeta_{1c} - \zeta_{2c}) = \xi_c \left[\frac{1+\beta^2}{2\beta} - \frac{1-\beta^2}{2\beta} \cos 2\, Z_{1c} \right], \tag{39}$$

where we have put

$$\xi_c^2 = \frac{\Gamma_{1c}}{\Gamma_{2c}}. \tag{40}$$

By way of example, we shall consider two very different situations: the one in which the two resonances are well-separated, the other in which they are superposed to each other, with a phase difference of nearly $\frac{1}{2}\pi$. For simplicity, we assume that a single channel is open; the index c may thus be dropped from all phase angles and channel widths (the latter being equal to the corresponding total widths). In harmony with the notation (40), we assume that Γ_1 is the smaller of the two widths. The scattering cross-section is proportional to the real part of the matrix element $1 - U_{cc}$ *i.e.* to

$$1 - \operatorname{Re} B_{cc} + q \sum_{n=1,2} \frac{\Gamma_n}{(E-E^n)^2 + \frac{1}{4}\Gamma_n^2} [\tfrac{1}{2}\Gamma_n \cos 2\zeta_n + (E-E_n) \sin 2\,\zeta_n].$$

Assume in the first place

$$|E_1 - E_2| \gg \Gamma_2 \geqslant \Gamma_1.$$

The equation (35) for β reduces to

$$\frac{2\beta}{1+\beta^2} \approx -\frac{\Gamma_2 \xi}{E_1 - E_2} \cos(\zeta_1 - \zeta_2), \tag{41}$$

or, by the first equation (39), and Eq. (37), to the equivalent equations

$$\frac{4\beta^2}{1-\beta^4} \approx -\frac{\Gamma_1}{E_1 - E_2} \sin 2Z_1 = \frac{\Gamma_2}{E_1 - E_2} \sin 2Z_2. \tag{42}$$

Assume further that little distortion of the Breit-Wigner shapes of the resonances is observed, which means that the angles ζ_1, ζ_2 are small, as well as the background phase. The equations (36) have then the approximate solution

$$Z_1 \approx \beta/\xi, \quad Z_2 \approx -\beta\xi,$$

which, inserted in Eq. (42), gives

$$\beta \approx -\tfrac{1}{2}\frac{\sqrt{\Gamma_1\Gamma_2}}{E_1-E_2}.$$

This expression, which obviously satisfies the inequality $|\beta| < \xi$, shows how the sign of β, which is a matter of convention, is fixed, in this case, by the relative position of the resonance poles labelled by the values of the index n.

Let us finally examine a situation characterized by the assumptions

$$E_1 = E_2, \quad Z_1 = \tfrac{1}{2}\pi-\zeta,$$

where ζ is a small quantity. Eq. (38) yields

$$Z_2 \approx -\beta^2\zeta,$$

and Eq. (35) becomes

$$\frac{2\beta}{1+\beta^2} = \frac{2\xi}{1+\xi^2}\sin(\zeta_1-\zeta_2),$$

or very nearly, by the second equation (39),

$$\frac{2\beta}{1+\beta^2} \approx \frac{2\xi}{1+\xi^2}\cdot\frac{\xi}{\beta},$$

showing that

$$\beta^2 \approx \xi^2,$$

in harmony with eq. (37). We have, by Eq. (34), $q \approx (1+\xi^2)/(1-\xi^2)$, and if the background phase is still smaller than ζ, *i.e.*, if we may identify the phases ζ_1, ζ_2 with Z_1, Z_2, we see that the resonance part of the cross-section (41) consists of a superposition of the two slightly distorted Breit-Wigner curves, the narrower one appearing as an indentation of the peak of the broader one, approximately as

$$\frac{1+\xi^2}{1-\xi^2}\cdot\frac{1}{2}\Gamma_2^2\left[\frac{-\xi^4}{(E-E_2)^2+\frac{1}{4}\xi^4\Gamma_2^2}+\frac{1}{(E-E_2)^2+\frac{1}{4}\Gamma_2^2}\right].$$

A similar situation (involving two channels) is discussed by Lynn (Ref. [6], p. 82).

APPENDIX

Relations between the matrices S and K

A.1. Transformation of the matrix K

Let us introduce the matrix d with elements

$$d_{k'k} = \delta_{k'k} - \tfrac{1}{2}i\sum_n \frac{u_{k'n}u_{nk}}{E-\mathscr{E}_n}. \qquad \text{(A.1)}$$

From Eq. (7), $S = (\delta + iK)(\delta - iK)^{-1}$, we see that this matrix d is the inverse of $\delta - iK$. If we therefore denote by $D_{kk'}$ the minor of $d_{k'k}$ in the determinant $\det d$, we obtain for iK the expression

$$iK = \delta - \frac{D}{\det \delta}, \tag{A.2}$$

showing that its poles are the roots of the equation

$$\det d = 0. \tag{A.3}$$

We shall now proceed to put the last equation into the form of a characteristic equation; a similar transformation of the minors $D_{kk'}$ will then lead us to a corresponding form for the matrix K.

A. 1. 1. Transformation of the determinant det d

Let us denote by $u_{(CN)}$ the rectangular matrix of C rows and N columns

$$u_{(CN)} = \|u_{kn}\| = \begin{Vmatrix} u_{11} \dots u_{1N} \\ \dots\dots\dots \\ u_{C1} \dots u_{CN} \end{Vmatrix}$$

and use a lower index (C) or (N) to indicate the rank of square matrices. We may write

$$\det d = \begin{vmatrix} d_{(C)} & u_{(CN)} \\ O_{(NC)} & \delta_{(N)} \end{vmatrix},$$

where $O_{(NC)}$ denotes a null matrix. Multiply now each column $(C+n)$ by $\frac{1}{2} i u_{nk'}/(E - \mathscr{E}_n)$, sum over n and add to column k'. With the notation

$$w_{(NC)} = \| \tfrac{1}{2} i u_{nk'}/(E - \mathscr{E}_n) \|,$$

the result may be written

$$\det d = \begin{vmatrix} \delta_{(C)} & u_{(CN)} \\ w_{(NC)} & \delta_{(N)} \end{vmatrix},$$

or

$$\det d = \frac{1}{\prod\limits_n (E - \mathscr{E}_n)} \begin{vmatrix} \delta_{(C)} & u_{(CN)} \\ \frac{1}{2} i \tilde{u}_{(NC)} & \chi_{(N)}(E) \end{vmatrix},$$

where $\chi_{(N)}(E)$ denotes the characteristic diagonal matrix

$$\chi_{(N)}(E) = \|(E - \mathscr{E}_n)\delta_{nn'}\|.$$

Finally, multiply, in the last expression for $\det d$, column k' by $-u_{k'n'}$, sum over k' and add to column $C+n'$. Defining the characteristic matric $\Delta_{(N)}(E)$ by

$$\Delta_{nn'}(E) = (E - \mathscr{E}_n)\delta_{nn'} - \tfrac{1}{2} i \sum_{k'} u_{nk'} u_{k'n'}, \tag{A.4}$$

we may write the result

$$\det d = \frac{1}{\prod_n (E-\mathcal{E}_n)} \begin{vmatrix} \delta_{(C)} & O_{(CN)} \\ \frac{1}{2}\, i\tilde{u}_{(NC)} & \Delta_{(N)}(E) \end{vmatrix},$$

or more simply

$$\det d = \frac{1}{\prod_n (E-\mathcal{E}_n)} \det \Delta(E). \tag{A.5}$$

The poles of the K-matrix are accordingly the roots of the characteristic equation

$$\det \Delta(E) = 0. \tag{A.6}$$

A. 1. 2. Transformation of the minor $D_{kk'}$

Let us denote by $M^{(k',k)}$, $M^{(k',\cdot)}$, $M^{(\cdot,k)}$ the matrix resulting from the matrix M by removal of row k' and column k, row k' only or column k only. With this notation, we may write

$$D_{kk'} = \begin{vmatrix} (-)^{k+k'} d^{(k',k)}_{(C-1)} & u^{(k',\cdot)}_{(C-1,N)} \\ O_{(N,C-1)} & \delta_{(N)} \end{vmatrix}$$

and by the same operations as for the determinant $\det d$, we obtain

$$D_{kk'} = \frac{1}{\prod_n (E-\mathcal{E}_n)} (-)^{k+k'} \begin{vmatrix} \delta^{(k',k)}_{(C-1)} & u^{(k',\cdot)}_{(C-1,N)} \\ \frac{1}{2}\, i\tilde{u}^{(\cdot,k)}_{(N,C-1)} & \chi_{(N)}(E) \end{vmatrix}.$$

We can restore the matrix $\frac{1}{2} i\tilde{u}_{(NC)}$ in the lower left field by inserting a k-th column whose last N elements are the $\frac{1}{2} iu_{nk}$ and the first $(C-1)$ elements are zeroes, and a k'-th row whose elements are all zeroes except the k-th one, which is unity; this transformation only alters the value of $D_{kk'}$ by a sign factor $(-)^{k+k'}$, and we may write

$$D_{kk'} = \frac{1}{\prod_n (E-\mathcal{E}_n)} \begin{vmatrix} \delta'_{(C)} & u'_{(CN)} \\ \frac{1}{2}\, i\tilde{u}_{(N,C)} & \chi_{(N)}(E) \end{vmatrix}.$$

In this formula, the matrix $\delta'_{(C)}$ is just the unit matrix if $k=k'$ and otherwise differs from the unit matrix by the elements $\delta'_{kk}=0$, $\delta'_{k'k}=1$, $\delta'_{k'k'}=0$; the matrix $u'_{(C,N)}$ differs from the matrix $u_{(C,N)}$ by its k'-th row elements all being zeroes. Performing now the same last operation on $D_{kk'}$ as on the determinant $\det d$, we get

$$D_{kk'} = \frac{1}{\prod_n (E-\mathcal{E}_n)} \begin{vmatrix} \delta'_{(C)} & O'_{(C,N)} \\ \frac{1}{2}\, i\tilde{u}_{(N,C)} & \Delta_{(N)}(E) \end{vmatrix},$$

where $O'_{(C,N)}$ denotes a null matrix in which the k'-th row is replaced by the one-row matrix

$u^k_{(1,N)} = ||u_{k1} \dots u_{kN}||$affected with a minus sign, and moreover, when $k \neq k'$, the k—th row is the matrix $u^k_{(1,N)}$. For $k = k'$ we therefore get immediately the simpler result

$$D_{kk} = \frac{1}{\prod_n (E - \mathscr{E}_n)} \begin{vmatrix} 1 & -u^k_{(1,N)} \\ \frac{1}{2}\, i\tilde{u}^k_{(N,1)} & \Delta_N(E) \end{vmatrix}.$$

For $k \neq k'$ let us replace the k-th row by the sum of the k-th and k'-th rows: this transforms the matrix $O'_{(C,N)}$ into a null matrix with the only exception of the k'-th row $-u^k_{(1,N)}$, and the matrix $\delta'_{(C)}$ into the unit matrix with the only exception of the elements $\delta'_{k'k} = 1$, $\delta'_{k'k'} = 0$. This gives

$$D_{kk'} = \frac{1}{\prod_n (E - \mathscr{E}_n)} \begin{vmatrix} 0 & -u^k_{(1,N)} \\ \frac{1}{2}\, i\tilde{u}^k_{(N,1)} & \Delta_{(N)}(E) \end{vmatrix}.$$

The two last formulae can be condensed into the following one:

$$D_{kk'} = \frac{1}{\prod_n (E - \mathscr{E}_n)} \begin{vmatrix} \delta_{kk'} & -u^k_{(1,N)} \\ \frac{1}{2}\, i\tilde{u}^k_{(N,1)} & \Delta_N(E) \end{vmatrix}. \tag{A.7}$$

A. 1. 3. Expression for the K-matrix

Combining the preceding results (A.5) and (A.7) we readily put the expression (A.2) for the matrix K into the form

$$K_{kk'} = \frac{1}{2 \det \Delta(E)} \begin{vmatrix} 0 & u^k_{(1,N)} \\ \tilde{u}^k_{(N,1)} & \Delta_{(N)}(E) \end{vmatrix}. \tag{A.8}$$

If ∇ denotes the inverse of the matrix Δ, we can write this expression as

$$K_{kk'} = -\tfrac{1}{2} \sum_{n,n'} u_{kn} \nabla_{nn'} u_{n'k'}. \tag{A.9}$$

A.2. Properties of the K-matrix

The symmetrical matrices Δ and ∇ can be diagonalized by a complex-orthogonal transformation $O = ||O_{n\mu}||$, with $\tilde{O}O = \delta$; calling the roots of Eq. (A.6) ε_μ, we get

$$\tilde{O}\Delta O = ||(E-\varepsilon_\mu)\delta_{\mu\mu'}||. \qquad \tilde{O}\nabla O = ||(E-\varepsilon_\mu)^{-1}\delta_{\mu\mu'}||. \tag{A.10}$$

Introducing also the transforms of the partial width parameters

$$v_{k\mu} = \sum_n u_{kn} O_{n\mu}, \quad v_{\mu k} = \sum_n \tilde{O}_{\mu n} u_{nk}, \tag{A.11}$$

we obtain a new representation of the matrix K,

$$K_{kk'} = -\tfrac{1}{2} \sum_\mu \frac{v_{k\mu} v_{\mu k'}}{E - \varepsilon_\mu}, \tag{A.12}$$

as a sum of contributions from each of its poles.

The last expression is convenient for ascertaining the reality properties of the poles and residues of the K-matrix. Since the elements $K_{kk'}$ are real for real values of E, the poles and residues are either real or pairs of complex conjugate quantities. It readily follows from this that the elements $K_{kk'}$ are R-functions [5], and consequently that all the poles ε_μ are real, as well as the products $v_{k\mu}v_{\mu k'}$. We have still to decide whether the $v_{k\mu}$ and $v_{\mu k}$ themselves are real or purely imaginary (since the sign of the matrix K had been chosen arbitrarily). The decision results from the condition that the imaginary parts of the resonance poles $\mathscr{E}_n = E_n - \frac{1}{2} i \Gamma_n$ must be negative, *i.e.* $\Gamma_n > 0$. Indeed, let us consider the sum of the roots $\sum \varepsilon_\mu$ of Eq. (A.6), *i.e.* on account of Eq. (A.4),

$$\sum_n [\mathscr{E}_n + \tfrac{1}{2} i \sum_{k'}] \cdot u_{nk'}u_{k'n} \qquad (A.13)$$

By inverting Eqs (A.11), we see that

$$\sum_{n,k'} u_{nk'}u_{k'n} = \sum_{\mu,k'} v_{\mu k'}v_{k'\mu},$$

which shows us that the left-hand side is a real quantity. The vanishing of the imaginary part of the expression (A.13) thus yields the relation

$$\sum_{n,k'} u_{nk'}u_{k'n} = \sum_n \Gamma_n,$$

from which we conclude that

$$\sum_{\mu,k'} v_{\mu k'}\, v_{k'\mu} > 0,$$

and that, therefore, the $v_{\mu k}$ and $v_{k\mu}$ must be real.

A.3. The transformation O

The preceding analysis exhibits the role of the complex-orthogonal transformation O in establishing a reciprocal relation between the resonance parameters $\mathscr{E}_n$, u_{kn} of the S-matrix and the analogous parameters ε_μ, $v_{k\mu}$ of the K-matrix. This relation may be concisely expressed in matrix notation. Let us denote, as above, by u^k the one-row matrix $||u_{k1} \ldots u_{kN}||$ and accordingly by $\tilde{u}^k$ the one-column matrix with the same elements; moreover, let us call $\mathscr{E}$ and ε the diagonal matrices $||\mathscr{E}_n\delta_{nn'}||$, $||\varepsilon_\mu\delta_{\mu\mu'}||$. From the first Eq. (A.10) and the expression (A.4) for the matrix Δ we derive

$$\mathscr{E} + \tfrac{1}{2} i \sum_k \tilde{u}^k u^k = O\varepsilon\tilde{O}, \qquad (A.14)$$

and we have in addition the inverted Eqs (A.11)

$$u^k = v^k\tilde{O}, \qquad \tilde{u}^k = O\tilde{v}^k. \qquad (A.15)$$

With the help of the last equations, we could also write Eq. (A.14) in the reciprocal form

$$\varepsilon + \tfrac{1}{2} i \sum_k \tilde{v}^k v^k = \tilde{O}\mathscr{E}O,$$

which shows that the resonance poles $\mathscr{E}_n$ could be found in terms of the poles and residues of the K-matrix as roots of the characteristic equation

$$\det\left\{(E-\varepsilon_\mu)\,\delta_{\mu\mu'} + \tfrac{1}{2} i \sum_k v_{\mu k}v_{k\mu'}\right\} = 0,$$

quite similar in form to the reciprocal characteristic equation (A.6)–(A.4). For the present purpose, however, this reciprocal aspect is not of interest: the Eqs (A.14), (A.15) are those directly adapted to the formulation of the unitarity conditions.

It is important to nte that a complex- orthogonal transformation like O can always be expressed as the product of a real transformation R of non-vanishing determinant and a hermitian transformation of the form $\gamma = \delta + i\beta$, where β is a real antisymmetrical matrix. Indeed, let us separate the real and imaginary parts of the matrix O:

$$O = R + iQ.$$

The orthogonality relations $\tilde{O}O = \delta$ become

$$\tilde{R}R - \tilde{Q}Q = \delta, \qquad \tilde{R}Q + \tilde{Q}R = 0. \tag{A.16}$$

From the first set of relations (A.16) one concludes that det $R \neq 0$ and that the matrix R therefore has an inverse; for $(\det R)^2 = \det(\delta + \tilde{Q}Q) = 1 + \Sigma$, where Σ is the sum of all principal minors of various ranks of the determinant det $\tilde{Q}Q$, all of which are non-negative. Let us accordingly introduce the matrix $\beta = QR^{-1}$ and write

$$O = \gamma R, \quad \gamma = \delta + i\beta. \tag{A.17}$$

The second set of orthogonality relations (A.16) immediately shows that the matrix β is antisymmetrical, $\beta + \tilde{\beta} = 0$, implying that the matrix γ is hermitian, $\tilde{\gamma} = \gamma^*$.

The orthogonality relations $\delta = O\tilde{O} = \gamma R\tilde{R}\tilde{\gamma} = (\delta + i\beta)\, R\tilde{R}(\delta - i\beta)$ show that the matrix β, and therefore also γ, commute with $R\tilde{R}$, and that the latter is accordingly the inverse of $\gamma\tilde{\gamma}$. Consequently $(\det \gamma)^2 = (\det R)^{-2}$, and since we have just seen that$(\det R)^2 > 1$, we find that the real determinant $C = \det \gamma = \det \tilde{\gamma}$ is less than unity in absolute value. Now, the general form of C as a function of the elements of the matrix β can be specified, according to well-known properties of skew determinants, as

$$C = 1 - \sum_{n_1 < n_2} (\beta_{n_1 n_2})^2 + \sum_{j=2}^{N'} (-)^j P^{(2j)}(\beta), \tag{A.18}$$

where $P^{(2j)}(\beta)$ denotes a function of the elements $\beta_{n_1 n_2}$ of degree $2j$, which has the form of a sum of squares of rational functions; the highest degree $2N'$ is either N or $N-1$ according as N is even or odd. Eq. (A.18) shows that there is a domain of values of the $(\beta_{n_1 n_2})^2 < 1$ in which $0 < C < 1$: it is this domain of possible determinations of the $\beta_{n_1 n_2}$ that is of interest in the usual conditions of our analysis.

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