SU (1,1) AS SOME KIND OF DYNAMICAL GROUP

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The problem of existence of a rotation invariant representation of SU(1,1) generating energy spectrum of a spinless particle subject to the action of central forces is investigated. Some assumptions based upon the 0(4,2) model of the nonrelativistic H-atom are made. Under these assumptions the construction of SU(1,1) representation is possible only for a very restricted class of potentials. Some of the obtained representations are used to find the energy spectrum of a relativistic, spinless particle in the attractive Coulomb potential.

Introduction

The considerations presented here are related to the Barut and Kleinert 0(4,2) model of the nonrelativistic H-atom [1, 2]. We focus our attention upon the role of the rotation invariant representation of the 0(2,1) subgroup of this group. This representation contains three generators $I_k (k = 1, 2, 3)$ all of them commuting with the orbital momentum operators. Two of them, say I_1 and I_2 are noncompact, the third I_3 is compact. One of the important features of the Barut–Kleinert model consists of the way in which the energy eigenfunctions are related to the eigenfunctions of the 0(2,1) generators. Namely, given the energy eigenfunction ψ_E corresponding to discrete or continuous part of the energy spectrum, one gets the eigenfunctions $\bar{\psi}_k$ of I_3 or I_1 respectively by applying the "mixing" operator *i.e.* $\bar{\psi}_{k(E)} = \exp[-i\varepsilon(E)I_2/\hbar]\psi_E$, where $\varepsilon(E)$ is the mixing parameter depending on energy eigenvalues E. The other important feature of the model is that the generator I_2 of the mixing operation is proportional to the dilatation generator $I_2 = c(\bar{r}\bar{p}+\bar{p}\bar{r})/2$. In Sec. 1 we attempt to answer the question of what potentials admit the existence of the rotation invariant representation of the SU(1,1) group (SU(1,1) is the covering group of the 0(2,1) group) if we conserve the two mentioned features of the Barut–Kleinert model. In Sec. 2 we demonstrate that some of the SU(1,1) representations found in Sec. 1 can be used to obtain the energy spectrum of the H-atom in the case of the nonrelativistic and relativistic Schrödinger equations. The last part of the work contains some concluding remarks.

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Section 1

We make the following assumptions:

1. The considered quantum mechanical system has its classical analogue. Stationary states ψ_E of the system are defined by the equation

a) $$\left[\frac{p^2}{2m} + V(r)\right] \psi_E = E\psi_E \tag{1}$$

in the nonrelativistic case, or

b) $$(p^2+m^2)\psi_E = [E-\varphi(r)]^2\psi_E \tag{2}$$

in the relativistic case (we put the velocity of light $c = 1$).

2. The generators I_k of the SU(1,1) representation are functions of the scalars $\bar{r}^2$, $\bar{r}\bar{p}$, $\bar{p}^2$ only. They fulfill the commutation relations

$$[I_1, I_2] = -i\hbar I_3; \quad [I_1, I_3] = -i\hbar I_2; \quad [I_2, I_3] = i\hbar I_1 \tag{3}$$

and generate a representation equivalent to the unitary representation.

3. One of the noncompact generators, say I_2, is proportional to the dilatation generator $I_2 = c(\bar{r}\bar{p}+\bar{p}\bar{r})/2$, where c is a real constant.

4. The stationary states are related to the eigenfunctions of I_3 or I_1 in the following way: if $\bar{\psi}_{k(E)} = \exp\,[-i\varepsilon(E)I_2/\hbar]\psi_E$ where $k(E)$ and $\varepsilon(E)$ are certain real functions of energy, then

a) $I_3\bar{\psi}_{k(E)} = k(E)\bar{\psi}_{k(E)}$ if ψ_E corresponds to a bound state,

b) $I_1\bar{\psi}_{k(E)} = k(E)\bar{\psi}_{k(E)}$ if ψ_E corresponds to a scattering state.

With these assumptions we shall prove that the only possible forms of potential are

a) $$V(r) = \pm\frac{\omega^2 r^2}{2} + \frac{\lambda}{r^2} \text{ or } V(r) = \frac{-\alpha}{r} \pm \frac{\lambda}{r^2} \tag{4}$$

in the nonrelativistic case ($\alpha, \omega^2 > 0$ and λ are some arbitrary real constants),

b) $$\varphi(r) = \frac{-\delta}{r}, \quad -\infty < \delta < +\infty \tag{5}$$

in the relativistic case.

To avoid the very complicated problem of calculating commutators of unknown functions of $\bar{r}^2$, $\bar{p}^2$ and $\bar{r}\bar{p}$ we may take the advantage of a well defined classical limit of our problem. Solving this classical problem we obtain the necessary condition for the existence of the corresponding quantum mechanical solutions. First of all we must explain the meaning of the assumptions 1–4 in the classical limit $\hbar \to 0$. The meaning of the assumptions 1–3 is obvious. Instead of 1–3 we have correspondingly:

1. The energy conservation formula takes the form

$$\frac{\bar{p}^2}{2m} + V(r) = E \tag{6}$$

in the nonrelativistic case, or

$$\bar{p}^2+m^2 = [E-\varphi(r)]^2 \tag{7}$$

in the relativistic case.

2. The clasical functions $\mathscr{I}_k(k = 1, 2, 3)$ of the scalars $\bar{r}^2$, $\bar{p}^2$ and $\bar{r}\bar{p}$ fullfil the Poisson bracket relations

$$\{\mathscr{I}_1,\mathscr{I}_2\} = -\mathscr{I}_3; \quad \{\mathscr{I}_1,\mathscr{I}_3\} = -\mathscr{I}_2; \quad \{\mathscr{I}_3,\mathscr{I}_2\} = -\mathscr{I}_1. \tag{8}$$

3. $\mathscr{I}_2 = c\bar{r}\bar{p}$.

The meaning of the assumption 4 in the classical limit can be found as follows. Let us consider bound states. Then according to the assumption 4 we have

$$I_3\bar{\psi}_{k(E)} = k(E)\bar{\psi}_{k(E)} \quad \text{and} \quad \bar{\psi}_{k(E)} = \exp\,[-i\varepsilon(E)I_2/\hbar]\psi_E,$$

where ψ_E is an energy eigenfunction. Thus we see that ψ_E fulfills the equation

$$\exp\,[i\varepsilon(E)I_2/\hbar]I_3 \exp\,[-i\varepsilon(E)I_2/\hbar]\psi_E = k(E)\psi_E. \tag{9}$$

Taking into account the commutation relations of I_k we can write this equation in the form

$$[\cosh \varepsilon(E)I_3 - \sinh \varepsilon(E)I_1]\psi_E = k(E)\psi_E. \tag{10}$$

Since this equation must be equivalent to the equation (1) or (2), we see that in the classical limit the equation

$$\cosh \varepsilon(E)\mathscr{I}_3 - \sinh \varepsilon(E)\mathscr{I}_1 = k(E) \tag{11}$$

must be equivalent to the nonrelativistic or relativistic energy conservation formulae for bound states. Similarly one gets that in the case of the scattering states the equation

$$-\sinh \varepsilon(E)\mathscr{I}_3 + \cosh \varepsilon(E)\mathscr{I}_1 = k(E) \tag{12}$$

must express the energy conservation formula for scattering states.

The Poisson bracket relations give us with $\mathscr{I}_2 = c\bar{r}\bar{p}$, a set of three differential equations for two functions $\mathscr{I}_1$ and $\mathscr{I}_3$

$$c\{\mathscr{I}_1, \bar{r}\bar{p}\} = -\mathscr{I}_3; \quad \{\mathscr{I}_1,\mathscr{I}_3\} = -c\bar{r}\bar{p}; \quad c\{\mathscr{I}_3, \bar{r}\bar{p}\} = -\mathscr{I}_1. \tag{13}$$

Since the scalars $\bar{p}^2$, $\bar{r}\bar{p}$, $\bar{r}^2$ can be expressed in terms of the variables r, pr and L^2, where p_r is the momentum conjugate to r, L^2 is the squared orbital momentum, and because $\{L^2, r\} = \{L^2, p_r\} = 0$ we can write (13) in the following way (we put $\mathscr{I}_k = \mathscr{I}_k(r, p_r, L^2)$):

$$c\left(\frac{\partial\mathscr{I}_1}{\partial r}r - \frac{\partial\mathscr{I}_1}{\partial p_r}p_r\right) = -\mathscr{I}_3; \; c\left(\frac{\partial\mathscr{I}_3}{\partial r}r - \frac{\partial\mathscr{I}_3}{\partial p_r}p_r\right) = -\mathscr{I}_1$$

$$\frac{\partial\mathscr{I}_1}{\partial r}\frac{\partial\mathscr{I}_3}{\partial p_r} - \frac{\partial\mathscr{I}_1}{\partial p_r}\frac{\partial\mathscr{I}_3}{\partial r} = -crp_r. \tag{14}$$

These equations can be easily solved. As a result we get

$$\mathscr{I}_1 = \frac{1}{2}\left[g(rp_r, L^2)p_r^\beta - \frac{\beta^{-2}(rp_r)^2 + A(L^2)}{g(rp_r, L^2)p_r^\beta}\right],$$

$$\mathscr{I}_3 = \frac{1}{2}\left[g(rp_r, L^2)p_r^\beta + \frac{\beta^{-2}(rp_r)^2 + A(L^2)}{g(rp_r, L^2)p_r^\beta}\right], \tag{15}$$

where $g(rp_r, L^2)$ is an arbitrary function of rp_r and L^2, $A(L^2)$ is an arbitrary function of L^2 only, and $\beta = c^{-1}$. Now we make the following important remark. Because the operators I_1 and I_3 are functions of the operators $\bar{p}^2$, $\overline{rp}_r$ and $\bar{r}^2$, the equation (10) is a differential equation in the position representation with the coefficients $\varepsilon(E)$ and $k(E)$. Since this equation must be equivalent in both cases of 1.a) and 1.b) to the second order differential equation for an arbitrary allowed E value, we see that I_1 and I_3 can be at most bilinear in p_r. Obviously this is true also in the classical limit, so we shall obtain some restrictions upon the form of the function $g(rp_r, L^2)$ in the formulas (15). Now for the sake of convenience we introduce the functions

$$\mathscr{I}_+ = \mathscr{I}_3 + \mathscr{I}_1; \quad \mathscr{I}_- = \mathscr{I}_3 - \mathscr{I}_1$$

which must be also bilinear in p_r. $\mathscr{I}_+ = g(rp_r, L^2)p_r^\beta$ satisfies this condition only if

$$g(rp_r, L^2) = [a(L^2) + b(L^2)rp_r + c(L^2)(rp_r)^2](rp_r)^{-\beta},$$

where $a(L^2)$, $b(L^2)$ and $c(L^2)$ are arbitrary functions of L^2. Then for $\mathscr{I}_-$ we get the result

$$\mathscr{I}_- = \frac{\beta^{-2}(rp_r)^2 + A(L^2)}{a + brp_r + c(rp_r)^2}\, r^\beta.$$

This expression is bilinear in p_r in one of the three following cases

1. $b = 0; \quad a = \alpha A; \quad c = \alpha\beta^{-2}$

then $\mathscr{I}_- = \alpha^{-1}r^\beta$ and $\mathscr{I}_+ = \alpha[A + \beta^{-2}(rp_r)^2]r^{-\beta}$.

2. $c = 0; \quad a^2\beta^{-2} + Ab^2 = 0$ so that

$$\mathscr{I}_- = \beta^{-2}b^{-2}(brp_r - a)r^\beta \quad \text{and} \quad \mathscr{I}_+ = ar^{-\beta}.$$

3. $b = c = 0$ so that

$$\mathscr{I}_- = a^{-1}[\beta^{-2}(rp_r)^2 + A]r^\beta; \quad \mathscr{I}_+ = ar^{-\beta}.$$

Correspondingly, we have

$$1. \quad \mathscr{I}_1 = \frac{1}{2}\{\alpha[A + \beta^{-2}(rp_r)^4]r^{-\beta} - \alpha^{-1}r^\beta\}; \; \mathscr{I}_2 = \beta^{-1}rp_r$$

$$\mathscr{I}_3 = \frac{1}{2}\{\alpha[A + \beta^{-2}(rp_r)^2]r^{-\beta} + \alpha^{-1}r^\beta\}. \tag{16}$$

$$2.\quad \mathscr{I}_1 = \frac{1}{2}\,[Aa^{-2}(brp_r - a)r^{\beta} + (a + brp_r)r^{-\beta}];$$

$$\mathscr{I}_3 = \frac{1}{2}\,[Aa^{-2}(-brp_r + a)r^{\beta} + (a + brp_r)r^{-\beta}];\ \mathscr{I}_2 = \beta^{-1}pr_r. \tag{17}$$

$$3.\quad \mathscr{I}_1 = \frac{1}{2}\,\{ar^{-\beta} - a^{-1}[\beta^{-2}(rp_r)^2 + A]r^{\beta}\}$$

$$\mathscr{I}_3 = \frac{1}{2}\,\{ar^{-\beta} + a^{-1}[\beta^{-2}(rp_r)^2 + A]r^{\beta}\};\ \mathscr{I}_2 = \beta^{-1}rp_r. \tag{18}$$

One can easily see that the cases 1 and 3 do not differ really. Namely, if we introduce in 3: $\mathscr{I}_1' = -\mathscr{I}_1$ and $\mathscr{I}_2' = -\mathscr{I}_2$, and then put $\lambda = -\beta$, $a = \alpha^{-1}$ we obtain case 1. Furthermore, we can at once exclude case 2 since it corresponds to the energy conservation formula linear in p_r. Thus in fact we have only to deal with the first possibility when $\mathscr{I}_1$, $\mathscr{I}_2$ and $\mathscr{I}_3$ take the form (16). Let us now consider the nonrelativistic and relativistic cases separately.

Nonrelativistic case

In the nonrelativistic case of bound states the classical equation

$$\cosh \varepsilon(E)\mathscr{I}_3 - \sinh \varepsilon(E)\mathscr{I}_1 = k(E) \tag{19}$$

according to the assumption 4 must be equivalent to the energy conservation formula

$$\frac{p_r^2}{2} + U_L(r) = E \tag{20}$$

where $U_L\ (r) = V(r) + \dfrac{L^2}{2r^2}$ and we have put $m = 1$.

Equation (19) can be transformed to the form

$$\mathscr{I}_+ + \mathscr{I}_- \exp\,[2\varepsilon(E)] = 2k(E)\ \mathrm{epx}\ \varepsilon(E) \tag{21}$$

where $\mathscr{I}_\pm = \mathscr{I}_3 \pm \mathscr{I}_1$.

Taking into account that $\mathscr{I}_+ = \alpha[A + \beta^{-2}(rp_r)^2]r^{-\beta}$ and $\mathscr{I}_- = \alpha^{-1}r^{\beta}$ we get

$$\alpha[A + \beta^{-2}(rp_r)^2]\ r^{-\beta} + e^{2\varepsilon(E)}\alpha^{-1}r^{\beta} = 2e^{\varepsilon(E)}k(E). \tag{22}$$

Now if we substitute $p_r^2 = 2[E - U_L(r)]$ into (22) we obtain the following identity condition

$$\alpha A r^{-\beta} + 2\alpha\beta^{-2}\,r^{2-\beta}[E - U_L(r)] + e^{2\varepsilon(E)}\alpha^{-1}\,r^{\beta} = 2e^{\varepsilon(E)}k(E). \tag{23}$$

This condition is satisfied only in two cases:

a) $\beta = 2$; $\varepsilon \neq \varepsilon(E)$ that is $\varepsilon = \mathrm{const}$ (we can put here $\varepsilon = 0$ as there is no a real difference between $\varepsilon = 0$ and $\varepsilon \neq 0$ in this case). Furthermore we must have the following equalities

$$U_L(r) = \frac{2A}{r^2} + \frac{2}{\alpha^2}\,r^2;\ k(E) = \frac{\alpha E}{4};\ A = \frac{L^2}{4} + \frac{\lambda}{4}, \tag{24}$$

where λ is an arbitrary real constant. Defining a new constant $\omega = \frac{2}{\alpha^2}$ we can write (24) in the form

$$U_L(r) = \frac{L^2}{2r^2} + \frac{\lambda}{r^2} + \frac{\omega^2 r^2}{2}\,;\; k(E) = E/2\omega\,;\; A = \frac{L^2}{4} + \frac{\lambda}{2}\,. \tag{25}$$

Correspondingly we then have

$$\mathscr{I}_+ = \frac{1}{2\omega}\left(p_r^2 + \frac{L^2+2\lambda}{r^2}\right);\; \mathscr{I}_- = \frac{\omega^2 r^2}{2}\,. \tag{26}$$

b) $\qquad \beta = 1;\; \varepsilon = \ln(-2\alpha^2 E)^{1/2};\; E < 0;\; A = L^2 + 2\gamma$

$$k(E) = \frac{\lambda}{2(-2\alpha^2 E)^{\frac{1}{2}}} \quad \text{and} \quad U_L(r) = \frac{A}{2r^2} - \frac{\lambda}{2\alpha r}\,, \tag{27}$$

where λ, γ are arbitrary real constants. Defining a new constant $\delta = \frac{\lambda}{2\alpha}$ we can write $U_L(r)$ in the form

$$U_L(r) = \frac{L^2}{2r^2} + \frac{\gamma}{r^2} - \frac{\delta}{r}\,. \tag{28}$$

Correspondingly we get

$$\mathscr{I}_+ = \alpha\left(rp_r^2 + \frac{2\gamma + L^2}{r}\right);\; \mathscr{I}_- = \alpha^{-1} r. \tag{29}$$

Since E must be negative we have an additional condition in the case $\delta < 0$. Namely, we must have in this case $L^2 + 2\gamma < 0$. Thus we see that for $\delta < 0$ bound states exist only for $L^2 < -2\gamma$. Similarly we have only the two possibilities $\beta = 1$ and $\beta = 2$ for the scattering states. However, there is an essential difference between the case $\beta = 1$ and $\beta = 2$ with respect to the whole energy spectrum of a considered physical system. In the case $\beta = 2$ we obtain for bound states the potential $U_{1L} = \frac{L^2}{2r^2} + \frac{\lambda}{r^2} + \frac{\omega^2 r^2}{2}$ and for the scattering states a different potential $U_{2L} = \frac{L^2}{2r^2} + \frac{\lambda}{r^2} - \frac{\omega^2 r^2}{2}$. These two potentials obviously correspond to different physical systems. U_{1L} corresponds to the system having only bound states, U_{2L} corresponds to the physical system having only scattering states. In the case $\beta = 1$ we obtain the same potential $U_L = \frac{L^2}{2r^2} + \frac{\gamma}{r^2} - \frac{\delta}{r}$ for bound and scattering states, and therefore these states belong to a one definite physical system. Bound and scattering states corresponds here to $E < 0$ and $E > 0$ respectively.

Relativistic case

Now the energy conservation formula

$$p^2 + m^2 = [E - \varphi(r)]^2 \tag{30}$$

must be equivalent to the equation

a) $\cosh\, \varepsilon(E)\mathscr{I}_3 - \sinh \varepsilon(E)\mathscr{I}_1 = k(E)$

for bound states, or

b) $-\sinh\, \varepsilon(E)\mathscr{I}_3 + \cosh\, \varepsilon(E)\mathscr{I}_1 = k(E)$

for the scattering states.

From equation a) and (16) one gets

$$\alpha[A+\beta^{-2}(rp_r)^2]\, r^{-\beta} + e^{2\varepsilon(E)}\alpha^{-1}\, r^{\beta} = 2e^{\varepsilon(E)}k(E). \tag{31}$$

On the other hand from (30) we have

$$p_r^2 = [E-\varphi(r)]^2 - \frac{L^2}{r^2} - m^2. \tag{30a}$$

Substituting (30a) into (31) we obtain the identity condition

$$\alpha \left\{A + \beta^{-2}r^2 \left[(E-\varphi)^2 - \frac{L^2}{r^2} - m^2\right]\right\} r^{-\beta} + e^{2\varepsilon(E)}\alpha^{-1}r^{\beta} = 2e^{\varepsilon(E)}k(E).$$

The above identity is satisfied only for $\beta = 1$; $m^2 - E^2 > 0$,

$$\varepsilon = \ln\,[\alpha^2(m^2-E^2)]^{\frac{1}{2}}; \;\; k(E) = \frac{\delta E}{(m^2-E^2)^{\frac{1}{2}}}; \;\; \varphi(r) = -\frac{\delta}{r}$$

and $A = L^2 - \delta^2$, where δ is an arbitrary real constant.

Correspondingly we get

$$\mathscr{I}_+ = \alpha \left(rp_r^2 + \frac{L^2-\delta^2}{r}\right); \;\; \mathscr{I}_- = \alpha^{-1}r. \tag{32}$$

In the case of relativistic scattering states we obtain in a similar way the identity

$$\alpha \left\{A + \beta^{-2}r^2 \left[(E-\varphi)^2 - \frac{L^2}{r^2} - m^2\right]\right\} r^{-\beta} - e^{2\varepsilon(E)}\alpha^{-1}r^{\beta} = 2e^{\varepsilon(E)}k(E), \tag{33}$$

which is satisfied also only for $\beta = 1$, $E^2 - m^2 > 0$,

$$A = L^2 - \delta^2,\; \varepsilon(E) = \ln\,[\alpha^2(E^2-m^2)]^{\frac{1}{2}}, \;\; k(E) = \frac{\delta E}{(E^2-m^2)^{\frac{1}{2}}}, \;\; \varphi(r) = -\frac{\delta}{r},$$

where δ is an arbitrary real constant. The form of $\mathscr{I}_+$ and $\mathscr{I}_-$ is the same.

Results of Section 1

The final results of the above considerations are the following. If the assumptions 1–4 are fulfilled in the classical limit then the potential $U_L(r)$ or $\varphi(r)$ can be only of the form

1. a) $$U_L(r) = \pm \frac{\omega^2 r^2}{2} + \frac{L^2+2\lambda}{2r^2}, \tag{34}$$

where $\omega^2 > 0$, λ are arbitrary real numbers, or

b) $$U_L(r) = -\frac{\delta}{r} + \frac{L^2+2\gamma}{2r^2}, \tag{35}$$

where δ, γ are arbitrary real numbers, in the nonrelativistic case; and:

2. $$\varphi(r) = -\frac{\delta}{r}, \tag{36}$$

where δ is an arbitrary real number, in the relativistic case. Now we shall write the Poisson bracket representation of the SU(1,1) algebra corresponding to each of the cases

1a) $$U_L(r) = \pm\frac{\omega^2r^2}{2} + \frac{L^2+2\lambda}{2r^2};\ k(E) = \frac{E}{2\omega};\ \varepsilon = 0$$

$$\mathscr{I}_1 = \frac{1}{2\omega}\left(\frac{p_r^2}{2} + \frac{L^2+2\lambda}{2r^2} - \frac{\omega^2r^2}{2}\right);\ \mathscr{I}_2 = \frac{1}{2}rp_r$$

$$\mathscr{I}_3 = \frac{1}{2\omega}\left(\frac{p_r^2}{2} + \frac{L^2+2\lambda}{2r^2} + \frac{\omega^2r^2}{2}\right). \tag{37}$$

1b) $$U_L(r) = -\frac{\delta}{r} + \frac{L^2+2\gamma}{2r^2};\ k(E) = \frac{\delta}{(2|E|)^{\frac{1}{2}}};$$

$$\varepsilon = \ln(2\alpha^2|E|)^{\frac{1}{2}};\ \mathscr{I}_1 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{2\gamma+L^2}{r^2}\right) - \frac{r}{\alpha}\right];$$

$$\mathscr{I}_2 = rp_r;\ \mathscr{I}_3 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{2\gamma+L^2}{r^2}\right) + \frac{r}{\alpha}\right]. \tag{38}$$

3) $$\varphi(r) = -\frac{\delta}{r};\ k(E) = \frac{\delta E}{|E^2-m^2|^{\frac{1}{2}}};\ \varepsilon = \ln[\alpha^2|E^2-m^2|]^{\frac{1}{2}}$$

$$\mathscr{I}_1 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{L^2-\delta^2}{r}\right) - \frac{r}{\alpha}\right];\ \mathscr{I}_2 = rp_r$$

$$\mathscr{I}_3 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{L^2-\delta^2}{r}\right) + \frac{r}{\alpha}\right]. \tag{39}$$

Having the above Poisson bracket representation of SU(1,1) we can easily construct corresponding operator representations. We write below the results

1a) $$U_L(r) = \pm\frac{\omega^2r^2}{2} + \frac{L^2+2\lambda}{2r^2};\ k(E) = \frac{E}{2\omega};\ \varepsilon = 0,$$

$$I_1 = \frac{1}{2\omega}\left[\frac{p_r^2}{2} + \frac{L^2+2\lambda}{2r^2} - \frac{\omega^2r^2}{2}\right];$$

$$I_2 = \frac{1}{4}(rp_r + p_rr);\ I_3 = \frac{1}{2\omega}\left[\frac{p_r^2}{2} + \frac{L^2+2\lambda}{2r^2} + \frac{\omega^2r^2}{2}\right], \tag{40}$$

where $p_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$. The sign "+" corresponds here to a physical system having only bound states and the sign "—" to a physical system having only scattering states.

1b) $$U_L = -\frac{\delta}{r} + \frac{L^2+2\gamma}{2r^2}\,;\quad k(E) = \frac{\delta}{(2|E|)^{\frac{1}{2}}}\,;\quad \varepsilon = \ln\,(2\alpha^2|E|)^{\frac{1}{2}}$$

$$I_1 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{L^2+2\gamma}{r}\right) - \frac{r}{\alpha}\right];\quad I_2 = rp_r;$$

$$I_3 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{L^2+2\gamma}{r}\right) + \frac{r}{\alpha}\right]. \tag{41}$$

2) $$\varphi(r) = -\frac{\delta}{r}\,;\quad k(E) = \frac{\delta E}{|E^2-m^2|^{\frac{1}{2}}}\,;\quad \varepsilon(E) = \ln\,(\alpha^2|E^2-m^2|)^{\frac{1}{2}}$$

$$I_1 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{L^2-\delta^2}{r}\right) - \frac{r}{\alpha}\right];\quad I_2 = rp_r$$

$$I_3 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{L^2-\delta^2}{r}\right) + \frac{r}{\alpha}\right]. \tag{42}$$

One can easily see that for I_k, $\varepsilon(E)$, $k(E)$ and $U_L(r)$ or $\varphi(r)$ given by (40), (41) or (42) the equation

$$[\cosh\,\varepsilon(E)I_3 - \sinh\,\varepsilon(E)I_1]\,\psi_E = k(E)\,\psi_E \tag{43}$$

for bound states, and

$$[-\sinh\,\varepsilon(E)I_3 + \cosh\,\varepsilon(E)I_1]\,\psi_E = k(E)\,\psi_E \tag{44}$$

for the scattering states, are equivalent to the corresponding nonrelativistic or relativistic Schrödinger equations.

Section 2

The SU(1,1) representations obtained in Sec. 1 can be used to find the energy spectra of the corresponding quantum mechanical systems. Let us consider the physically most interesting examples of the nonrelativistic or relativistic spinless particle subject to the action of an attractive Coulomb force.

1. Nonrelativistic case

In this case one has $U_L(r) = \frac{-\delta}{r} + \frac{L^2}{2r^2}$; $\delta > 0$ and according to (41)

$$I_1 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{L^2}{r}\right) - \frac{r}{\alpha}\right];\quad I_2 = rp_r;\quad k(E) = \frac{\delta}{(2|E|)^{\frac{1}{2}}}$$

$$I_3 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{L^2}{r}\right) + \frac{r}{\alpha}\right]. \tag{45}$$

The representation (45) is equivalent to a unitary representation. Namely, if I_k are given by (45) then $I'_k = r^{-\frac{1}{2}} I_k r^{\frac{1}{2}}$ are already hermitian. To find the spectrum of I_3 or I_1 we must calculate the Casimir operator $W^2 = \hbar^{-2}(I_1^2+I_2^2-I_3^2)$ of the representation (45). After simple calculation one gets

$$W^2 = -\frac{L^2}{\hbar^2} = -l(l+1). \tag{46}$$

For this value of W^2 we have two possible unitary representations of SU(1,1) namely $D^+(l+1)$ and $D^-(l+1)$ [4,5]. The corresponding eigenvalues i_3 of I_3 are

$$\begin{aligned} &\text{for } D^-(l+1); \quad i_3 = (-l-1-n')\hbar; \quad n' = 0, 1, 2\ldots \\ &\text{for } D^+(l+1); \quad i_3 = (l+1+n')\hbar; \quad n' = 0, 1, 2\ldots \end{aligned} \tag{47}$$

Since $\delta > 0$ and consequently $k(E) > 0$ we must choose the representation $D^+(l+1)$. Then from the equality $k(E) = i_3$ we obtain for $E < 0$

$$\delta(-2E)^{-\frac{1}{2}} = (l+1+n')\hbar \quad \text{and} \quad E = \frac{-\delta^2}{2\hbar^2(n'+l+1)^2} \quad . \tag{48}$$

in agreement with the usual formulae for the bound states.

In the case of the scattering states $k(E)$ is related to the eigenvalues i_1 of I_1, so that $k(E) = i_1$. Since for $D^+(l+1)$, $0 < i_1^2 < \infty$ we obtain the proper energy spectrum also for the scattering states

$$E = \frac{\delta^2}{2\hbar^2 i_1^2}\,; \; 0 < E < \infty. \tag{49}$$

2. Relativistic case

In the relativistic case of the attractive Coulomb interaction we have $\varphi(r) = \frac{-\delta}{r}$; $\delta > 0$ and according to the formula (42)

$$I_1 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{L^2-\delta^2}{r}\right) - \frac{r}{\alpha}\right]; \; I_2 = rp_r; \; k(E) = \frac{\delta E}{|E^2-m^2|^{\frac{1}{2}}}$$

$$I_3 = \frac{1}{2}\left[\alpha\left(rp_r^2 + \frac{L^2-\delta^2}{r}\right) + \frac{r}{\alpha}\right]. \tag{50}$$

This SU(1,1) representation is identical to the representation (45) with the only exception that now L^2 is replaced by the difference $L^2-\delta^2$. Calculation of the Casimir operator $W^2 = \hbar^{-2}(I_1^2+I_2^2-I_3^2)$ gives the result $W^2 = -\hbar^{-2}(L^2-\delta^2) = -l(l+1) + \frac{\delta^2}{\hbar^2}$. This value of W^2 corresponds to the unitary irreducible representations

$$D^+\left(\frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{\delta^2}{\hbar^2}}\right) \quad \text{and} \quad D^-\left(\frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{\delta^2}{\hbar^2}}\right).$$

The eigenvalues i_3 of I_3 are the following
for

$$D^+\left(\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{\delta^2}{\hbar^2}}\right);\ i_3=\left[\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{\delta^2}{\hbar^2}}+n'\right];\ n'=0,1,2.$$

for

$$D^-\left(\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{\delta^2}{\hbar^2}}\right);\ i_3=\left[-\frac{1}{2}-\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{\delta^2}{\hbar^2}}-n'\right];\ n'=0,1,2\ldots$$

Let us consider now the bound states $E^2 < m^2$. Then $i_3 = \frac{\delta E}{\sqrt{m^2-E^2}}$. Since $\delta > 0$ and $E > 0$ then $i_3 > 0$ so that D^- must be excluded. Consequently, one gets

$$i_3=\left[\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{\delta^2}{\hbar^2}}+n'\right]\hbar=\frac{\delta E}{\sqrt{m^2-E^2}}$$

and

$$E=m\left[1+\frac{\delta^2}{(s+n')^2\hbar^2}\right]^{-\frac{1}{2}} \tag{51}$$

where $s=\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{\delta^2}{\hbar^2}}$. The formula (51) is the usual formula for the energy levels of bound states given by the relativistic Schrödinger equation in the Coulomb potential. For scattering states $k(E) = i_1$ and we obtain

$$E=m\left(1-\frac{\delta^2}{i_1^2}\right)^{-\frac{1}{2}}.$$

Since for $D^+\left(\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{\delta^2}{\hbar^2}}\right)$, $\delta^2 < i_1^2 < \infty$ then, as is expected, $m < E < \infty$.

Final conclusions

As one can see the physical assumptions 2–4, expressed in group-theoretical language, give for the equations for the stationary states accepted in the assumption 1, strong restrictions upon the form of interaction. Thus in the case of the relativistic, spinless particle we have got the result, that the only possible interaction is given by the Coulomb potential. The SU(1,1) representation obtained in our work for this potential strictly corresponds to the rotation invariant representation of the 0(2,1) subgroup of the 0(4,2) dynamical group used by Barut and Kleinert to describe the nonrelativistic H-atom. The results which can be obtained for the nonrelativistic H-atom in the framework of this 0(2,1) representation, can be obtained in our case with the help of suitable representations of SU(1,1) which do not belong to the ordinary (singlevalued) representations of 0(2,1) [4,5]. Thus using the representations of SU(1,1) we can find the energy levels and calculate the probabilities of the transitions which do not change the orbital momentum eigenvalues in the case of the relativistic spinless particle placed in the Coulomb potential. The interesting question is if the group SU(1,1) can be extended to include all electromagnetic transitions operators and if a full dynamical group, similar to the 0(4,2) group for the nonrelativistic H-atom can be constructed. However, one may expect serious difficulties here, since in the relativistic case the energy levels are not degenerate with respect to the orbital momentum eigenvalues. Another interesting problem is to study the possibility of constructing the rotation invariant group representation similar to our SU(1,1) in the case of the relativistic particles with spin. Investigating the last problem we have already obtained some results in the case of a particle described by the Dirac equation. They will be published elsewhere.

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