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ON WEYL'S THEORY OF GRAVITATION AND ELECTROMAGNETISM

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A new interpretation of Weyl's theory of gravitation and electromagnetism is proposed, in which it appears related to Rainich's Already Unified Field Theory. Some of the equations of the latter are derived by subjecting Weyl's action integrals to variations in the metric tensor $g_{\mu\nu}$ and in the tensor $S^{\lambda}_{\mu\nu}$, obtained by subtracting an affine connection $\Gamma^{\lambda}_{\mu\nu}$ which defines parallel transfer of vectors in the presence of matter, from the Riemannian Christoffel brakets of General Relativity. Matter therefore, appears as a relative tilting of vectors at different points of a Riemannian manifold of four dimensions, which are regarded as parallel by a relativistic observer.

1. *Introduction*

It is commonly held that Weyl's unified field theory [1], and its generalisation due to Edington [2], are based on a true extension of Riemannian geometry. So far, no unified field teory has gained acceptance; only recently a suggestion has been made how to test experimentally at least that class of theories which involves a space-time torsion [3]. The experiment, if performed, should discriminate between the latter, and between theories of Weyl's type, though without proving either. It is important, therefore, to have a clear picture of the logical hypotheses on which various proposals are founded.

We suggest in this article that Weyl's theory can be established without requiring the space-time manifold to be non-Riemannian. We need only to interpret in a suitable way various quantities which appear naturally within the context of a four-dimensional Riemann space V_4. In this sense, Weyl's theory seems to bear a closer relationship to Rainich's *Already Unified Field Theory* [4] than to the later work of Einstein, Schrödinger and others (*e. g.* [5, 6]. In Einstein's theory, the meaning of a Riemannian geometry is obscured by the ultimate necessity of relating mathematics to physics.

Actually, we are aiming at something more. Eddington observes (*loc. cit.*, p. 221) that, a purely gravitational field apart, matter is to be described a tensor $K^{\lambda}_{\mu\nu}$ (throughout this article, Greek indices go from 1 to 4, the usual summation convention is observed, and tensor indices are raised and lowered with the help of a symmetric "metric" tensor $g_{\mu\nu}$).

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The tensor K is supposed to represent a physical "thing", but is essentially only a shorthand notation for the covariant derivative of $g_{\mu\nu}$, formed with the help of an affine connection $\Gamma^{\lambda}_{\mu\nu}$ · $\Gamma^{\lambda}_{\mu\nu}$ itself is to be determined from the field equations derived from a variational action principle. In this search for an immediate electromagnetic interpretation, Weyl put

$$K^{\lambda}_{\mu\nu} = g_{\mu\nu}\varphi^{\lambda},$$

where φ^{λ} was to be the four-vector potential.

Eddington refrained from this simplification at the start of his work but fell back on it when it came to the derivation of the field equations. In the present article, we shall vary the action invariant with respect to an unspecified tensor K[1] (and with respect to the metric) in order to obtain as general a form of the theory as possible. It is perhaps surprising that even then we are forced to assume that the potential is proportional to another vector which must be interpreted as an electric current. This is the relation derived by Weyl; the alternative is to admit a very peculiar electrodynamics. We write, with Eddington.

$$g_{\mu\nu\,;\,\lambda} = g_{\mu\nu,\,\lambda}\Gamma^{\alpha}_{\mu\lambda} - g_{\alpha\nu} - \Gamma^{\alpha}_{\nu\lambda}g_{\mu\alpha} = K_{\mu\nu\lambda} \tag{1}$$

where comma denotes ordinary partial differentiation.
Hence

$$K_{\mu\nu\lambda} = K_{\nu\mu\lambda}$$

We can regard (1) as an equation defining either the "matter" tensor $K_{\mu\nu\lambda}$ of the affine connection $\Gamma^{\lambda}_{\mu\nu}$, assumed to be symmetric so that infinitesimal parallelograms may close. The object of introducing $\Gamma^{\lambda}_{\mu\nu}$ is, precisely, to define vector parallelism.

As we choose different connections, we agree to regard different vectors at some point Q of the manifold, as parallel to a given vector at another point P. It follows from (1) that

$$\Gamma^{\lambda}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} - S^{\lambda}_{\mu\nu}\,, \tag{2}$$

where

$$S^{\lambda}_{\mu\nu} = \frac{1}{2}\, g^{\lambda\sigma}(K_{\sigma\nu\mu} + K_{\mu\sigma\nu} - K_{\mu\nu\sigma}), \tag{3}$$

and

$$\left\{ {\lambda \atop \mu\nu} \right\} = \frac{1}{2}\, g^{\lambda\sigma}(g_{\sigma\nu,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}),$$

are the usual Christoffel brackets of the second kind.

The Ricci tensor

$$^{*}R_{\mu\nu} = -\Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\sigma}_{\mu\varrho}\Gamma^{\varrho}_{\sigma\nu} - \Gamma^{\varrho}_{\mu\nu}\Gamma^{\sigma}_{\varrho\sigma}, \tag{4}$$

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[1] Or, rather, with respect to a combination S of its components.

has, in general, a nonvanishing, skew symmetric part

$$*R_{\underset{\vee}{\mu\nu}} = -*R_{\underset{\vee}{\nu\mu}} = \frac{1}{2}\,(S^{\sigma}_{\nu\sigma,\mu} - S^{\sigma}_{\mu\sigma,\nu}). \tag{5}$$

But, from (2), $S^{\lambda}_{\mu\nu}$ is the difference of two connections, and therefore is a tensor. Hence

$$\varphi_{\mu} = \frac{1}{2}\,S^{\sigma}_{\mu\sigma} \tag{6}$$

is a vector whose curl is $*R_{\underset{\vee}{\mu\nu}}$. It follows that

$$*R_{\underset{\vee}{\mu\nu},\lambda} + *R_{\underset{\vee}{\nu\lambda},\mu} + *R_{\underset{\vee}{\lambda\mu},\nu} = 0, \tag{7}$$

and we can identify $*R_{\mu\nu}$ with the electromagnetic field intensity tensor $f_{\mu\nu}$. We may note that when $K_{\mu\nu\lambda} = g_{\mu\nu}k_{\lambda}$, we get $k_{\lambda} = \varphi_{\lambda}$. Also, φ_{λ} is determined by (5) only up to the addition of a gradient:

$$\varphi'_{\lambda} = \frac{1}{2}\,S^{\sigma}_{\lambda\sigma} + \lambda_{\lambda}$$

would have done just as well.

Having got $f_{\mu\nu}$ we define a current vector density $\mathfrak{J}^{\mu}$ by

$$\mathfrak{f}^{\mu\nu}{}_{,\nu} \equiv \frac{\partial}{\partial x^{\nu}}\,(\sqrt{-g}\,f^{\mu\nu}) = \mathfrak{J}^{\mu}, \tag{8}$$

where $g = \det\,(g_{\mu\nu})$. Furthermore, from (4),

$$*R_{\mu\nu} = R_{\mu\nu} + S^{\sigma}_{\mu\nu|\sigma} - S^{\sigma}_{\mu\sigma|\nu} + S^{\sigma}_{\mu\varrho}S^{\varrho}_{\sigma\nu} - S^{\sigma}_{\mu\nu}S^{\varrho}_{\sigma\varrho}, \tag{9}$$

where $R_{\mu\nu}$ is the Ricci tensor formed with the help of $\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\}$, as in General Relativity; the stroke denotes a covariant derivative with $\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\}$ as the affine connection. Thus we still have

$$g_{\mu\nu|\lambda} = 0.$$

We shall also use the tensor

$$P_{\mu\nu} = *R_{\mu\nu} - R_{\mu\nu}. \tag{10}$$

In particular, the symmetric part of P

$$P_{\underline{\mu\nu}} = *R_{\underline{\mu\nu}} - R_{\underline{\mu\nu}}, \quad \text{and} \quad P_{\underset{\vee}{\mu\nu}} = *R_{\underset{\vee}{\mu\nu}} = f_{\mu\nu}.$$

All the above formulae are well known from Eddington's book; we recall them here because of their use in the variations which follow.

We adopt as an action invariant, first,

$$\mathscr{H}_1 = \sqrt{-g}\,(*R^2 + \beta f_{\mu\nu}f^{\mu\nu}), \tag{11}$$

where β is a constant and

$$*R = g^{\mu\nu}*R_{\mu\nu} = g^{\mu\nu}*R_{\underline{\mu\nu}}, \tag{12}$$

secondly, we shall take

$$\mathscr{H}_2 = \sqrt{-g}\,{}^*R_{\mu\nu}{}^*R^{\mu\nu}. \tag{13}$$

For the sake of convenience we shall vary $\mathscr{H}_1$, and $\mathscr{H}_2$ with respect to $\mathfrak{g}^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}$ and $S^{\lambda}_{\mu\nu}$ rather then $K_{\mu\nu\lambda}$. There can be no loss of generality until field equations are actually derived and interpreted.

We may note, finally, that Einstein [7] proposed a theory in which $\Gamma^{\lambda}_{\mu\nu}$ was used as the arbitrary variable. Ours is an analogous theory but it seems that some of the alleged artificiality of Einstein's proposal is eliminated.

2. *Derivation of the field equations: the case* $\mathscr{H}_1$

If we adopt the interpretation of

$$f_{\mu\nu} = {}^*R_{\underset{\vee}{\mu\nu}} = \varphi_{\nu,\mu} - \varphi_{\mu,\nu}, \tag{14}$$

as the electromagnetic field, it follows from equation (12) that we cannot use $\sqrt{-g}\,{}^*R$ as the action density. The electromagnetic quantities simply disappear from it althoug the density remains dependent on $S^{\lambda}_{\mu\nu}$. We add the term $\beta f_{\mu\nu} f^{\mu\nu}$ precisely in order to preserve electromagnetic terms. A density

$$\sqrt{-g}\,({}^*R + \beta f_{\mu\nu} f^{\mu\nu}) \tag{15}$$

could be used, but it would appear dimensionally unbalanced unless this were remedied by a suitable choice of β. Such densities have been used (*e. g.* [8]), leading to theories essentially distinct from Weyl's. Since we are investigating in this article the theory of Weyl, we adopt H_1 given by (11), without further ado.

Consider the stationary action principle

$$\delta \int_V \mathscr{H}_1\, d\tau = 0 \tag{16}$$

where $d\tau = dx^1\, dx^2\, dx^3\, dx^4$. We assume that all integrated quantities (triple integrals) vanish at the boundary of the region V in which the variation is being contemplated.

We can write (16) in the form

$$I_1 + I_2 = 0,$$

where

$$I_1 = 2\int d\tau\, (\mathfrak{g}^{\mu\nu}\,{}^*R\,\delta\,{}^*R_{\mu\nu} + \beta \mathfrak{f}^{\mu\nu}\,\delta f_{\mu\nu}).$$

and

$$I_2 = \int dr \left[2({}^*R\,{}^*R_{\mu\nu} + \beta g^{\mu\beta} f_{\mu\alpha} f_{\nu\beta}) \left(\delta^{\mu}_{\varrho}\delta^{\nu}_{\sigma} - \frac{1}{2}\, g^{\eta\nu} g_{\varrho\sigma} \right) + \frac{1}{2}\, H_1 g_{\varrho\sigma} \right] \delta\mathfrak{g}^{\varrho\sigma}$$

and

$$H_1 \sqrt{-g} = \mathscr{H}_1$$

Consider first the variation $\delta S^{\lambda}_{\mu\nu}$ in the tensor $S^{\lambda}_{\mu\nu}$: it comes exclusively from I_1. We have

$$\delta^* R_{\mu\nu} = \delta R_{\mu\nu} + S^{\alpha}_{\alpha\nu|\alpha} - \delta S^{\alpha}_{\mu\nu|\alpha} + S^{\beta}_{\alpha\nu}\delta S^{\alpha}_{\mu\beta}$$
$$+ S^{\alpha}_{\mu\beta}\delta S^{\beta}_{\mu\nu} - S^{\beta}_{\alpha\beta}\delta S^{\alpha}_{\mu\nu} - S^{\alpha}_{\mu\nu}\delta S^{\beta}_{\alpha\beta},$$

and

$$\mathfrak{f}^{\mu\nu} df_{\mu\nu} = f^{\mu\nu}(\delta S^{\alpha}_{\nu\alpha})_{,\mu},$$

where

$$\mathfrak{f}^{\mu\nu} \equiv \sqrt{-g}\, f^{\mu\nu}.$$

It can be shown easily that

$$\int \mathfrak{g}^{\mu\nu} {}^*R \delta S^{\alpha}_{\mu\nu|\alpha} d\tau = -\int \mathfrak{g}^{\mu\nu} {}^*R_{,\alpha} \delta S^{\alpha}_{\mu\nu} d\tau - \int ({}^*R S^{\alpha}_{\mu\nu})_{|\alpha} \delta \mathfrak{g}^{\mu\nu} d\tau,$$

and

$$\int \mathfrak{g}^{\mu\nu} {}^*R \delta S^{\alpha}_{\mu\alpha|\nu} d\tau = -\int \mathfrak{g}^{\mu\nu} {}^*R_{,\nu} \delta S^{\alpha}_{\mu\alpha} d\tau - \int ({}^*R S^{\alpha}_{\mu\beta})_{,\nu} \delta \mathfrak{g}^{\mu\nu} d\tau.$$

All that we require is to remember that covariant and partial differentiations behave exactly in the same way with respect to the integration by parts of invariant integrals; we must also assume, as is usual, that δ, $\dfrac{\partial}{\partial x^{\mu}}$ and the integration operator, commute with each other.

The result of equating to zero the co-factor $\delta S^{\alpha}_{\mu\nu}$ is then

$$-(\mathfrak{g}^{\mu\nu}R)_{|\alpha} + \frac{1}{2}(\mathfrak{g}^{\mu\sigma} {}^*R)_{|\sigma}\delta^{\nu}_{\alpha} + \frac{1}{2}(\mathfrak{g}^{\nu\sigma} {}^*R)_{|\sigma}\delta^{\mu}_{\alpha} + \mathfrak{g}^{\mu\sigma}S^{\nu}_{\alpha\sigma} {}^*R + \mathfrak{g}^{\nu\sigma}S^{\mu}_{\alpha\sigma} {}^*R - \mathfrak{g}^{\mu\nu}S^{\beta}_{\alpha\beta} {}^*R -$$

$$-\frac{1}{2}\mathfrak{g}^{\varrho\sigma}(S^{\mu\varrho}_{\sigma}\delta^{\nu}_{\alpha} + S^{\nu}_{\varrho\tau}\delta^{\mu}_{\alpha}) {}^*R + \frac{\beta}{2}(\mathfrak{J}^{\mu}\delta^{\nu}_{\alpha} + \mathfrak{J}^{\nu}\delta^{\mu}_{\alpha}) = 0. \tag{17}$$

Contracting this equation over ν and α, we obtain

$$(\mathfrak{g}^{\mu\sigma} {}^*R)_{|\sigma} = \mathfrak{g}^{\mu\sigma} {}^*R_{,\sigma} = \mathfrak{g}^{\varrho\sigma}S^{\mu}_{\varrho\sigma} {}^*R - \frac{5}{3}\beta\mathfrak{J}^{\mu}, \tag{18}$$

so that, from (17),

$$\mathfrak{g}^{\mu\nu} {}^*R_{,\alpha} = (\mathfrak{g}^{\mu\sigma}S^{\nu}_{\alpha\sigma} + \mathfrak{g}^{\nu\sigma}S^{\mu}_{\alpha\sigma} - \mathfrak{g}^{\mu\nu}S^{\sigma}_{\alpha\sigma}) - \frac{\beta}{3}(\mathfrak{J}^{\mu}\delta^{\nu}_{\alpha} + \mathfrak{J}^{\nu}\delta^{\mu}_{\alpha}). \tag{19}$$

Therefore,

$${}^*R_{,\alpha} = \frac{1}{2} S^{\sigma}_{\alpha\sigma} {}^*R - \frac{\beta}{6} j_{\mu}. \tag{20}$$

where

$$\mathfrak{J}_{\alpha} = j_{a}\sqrt{-g}$$

The next step is to consider either the integrability condition

$${}^*R_{,\alpha\beta} - {}^*R_{,\beta\alpha} = 0,$$

of equation (20), or the equation

$$\mathfrak{J}^{\mu}_{,\mu} = 0$$

of conservation of electric current which holds in virtue of the definition (8). The former is more instructive and gives

$$\frac{6^*R}{\beta} f_{\alpha\beta} = j_{\alpha,\beta} - j_{\beta,\alpha} + j_\alpha \varphi_\beta - j_\beta \varphi_\alpha . \tag{21}$$

This represents an addition to the laws of the electromagnetic field. It can be eliminated if we take with Weyl (and, incidentally, also with Dirac, Ref. [9])

$$\varphi_\alpha = k j_\alpha \tag{22}$$

where

$$k = -\beta/6^*R, \tag{23}$$

is a constant so that $^*R$ must be constant, too. This is precisely what happens in Weyl's theory where a guaging equation

$$^*R_{\mu\nu} = -\frac{\beta}{24k} g_{\eta\nu}, \tag{24}$$

is postulated.

To find the result of varying $\mathfrak{g}_{\mu\nu}$ we observe that since (Palatini)

$$\delta R_{\mu\nu} = \left(\delta \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}\right)_{|\alpha} + \left(\delta \begin{Bmatrix} \alpha \\ \mu\alpha \end{Bmatrix}\right)_{|\nu},$$

the contribution from this term may be written as

$$\mathfrak{M}^{\mu\nu}_{\alpha} \delta\delta \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}.$$

where

$$\mathfrak{M}^{\alpha}_{\mu\nu} \equiv 2\mathfrak{g}^{\mu\nu *}R_{,\alpha} - \mathfrak{g}^{\mu\sigma *}R_{,\sigma}\delta^{\alpha}_{\nu} - \mathfrak{g}^{\nu\sigma *}R_{,\sigma}\delta^{\mu}_{\alpha}.$$

This vanishes when $^*R$ is a constant.

Therefore, equating to zero the co-factor of $\sigma\mathfrak{g}^{\varrho\sigma}$ from I_1 and I_2, we get

$$-2(^*RS^{\alpha}_{\varrho\sigma})_{|\alpha} + (^*RS^{\alpha}_{\mu\alpha})_{|\nu} + (^*RS^{\alpha}_{\nu\alpha})_{|\mu} + 2(^*R^*R_{\mu\nu} + \beta g^{\alpha\beta} f_{\mu\alpha} f_{\nu\beta}) \times$$
$$\times \left(\delta^{\mu}_{\varrho}\delta^{\nu}_{\sigma} - \frac{1}{2} g^{\mu\nu} g^{\varrho\sigma}\right) + \frac{1}{2} H_1 g_{\varrho\sigma} = 0$$

or

$$R_{\varrho\sigma} - \frac{1}{4} R g_{\varrho\sigma} - \frac{1}{4} P g_{\varrho\sigma} - \frac{\beta}{^*R} E_{\varrho\sigma} + S^{\alpha}_{\varrho\beta} S^{\beta}_{\varrho\sigma} - S^{\alpha}_{\tau\sigma} S^{\beta}_{\alpha\beta} = 0, \tag{25}$$

where

$$E_{\varrho\tau} = f_{\alpha\beta} f^{\beta}_{\varrho} + \frac{1}{4} g_{\varrho\sigma} f_{\mu\nu} f^{\mu\nu},$$

is the electromagnetic energy-stress-momentum tensor.

Hence

$$P = g^{\varrho\sigma}(S^{\alpha}_{\varrho\beta}S^{\beta}_{\alpha\sigma} - S^{\alpha}_{\varrho\sigma}S^{\beta}_{\alpha\beta})$$

and (25) may be written in the form

$$R_{\varrho\sigma} - \frac{1}{4}Rg_{\varrho\sigma} = \frac{\beta}{{}^*R}E_{\varrho\sigma} - (S^{\alpha}_{\mu\beta}S^{\beta}_{\beta\nu} - S^{\alpha}_{\mu\nu}S^{\beta}_{\alpha\beta})\left(S^{\mu}_{\varrho}S^{\nu}_{\sigma} - \frac{1}{4}g^{\mu\nu}g_{\varrho\sigma}\right). \quad (26)$$

3. *Derivation of the field equations: the Case* $\mathscr{H}_2$

When $\mathscr{H}_2$ is chosen as the action density in place of $\mathscr{H}_1$, we obtain different field equations unless we should assume, as in Weyl's theory, that

$${}^*R_{\mu\nu} = \lambda g_{\mu\nu}$$

where λ can be a scalar function of position. It $\beta = 4$, then

$$\mathscr{H}_1 = 4\mathscr{H}_2.$$

Without this assumption, we proceed as follows:

$$\delta\int H_1\sqrt{-g}\,d\tau = \delta\int R^{\mu\nu*}R^{\mu\nu}\sqrt{-g}\,d\tau$$

$$= \int (H_2\delta\sqrt{-g} + \sqrt{-g}\,{}^*R_{\alpha\beta}{}^*R_{\mu\nu}(g^{\nu\beta}\delta g^{\mu\alpha} + g^{\mu\alpha}\delta g^{\nu\beta}) + 2\sqrt{-g}\,{}^*R^{\mu\nu}\delta^*R_{\mu\nu})d\tau = 0.$$

Let

$${}^*\mathfrak{R}^{\mu\nu} = \sqrt{-g}\,{}^*R^{\mu\nu},$$

so that

$$\int 2{}^*\mathfrak{R}^{\mu\nu}\delta^*R_{\mu\nu}d\gamma = \int 2{}^*\mathfrak{R}^{\mu\nu}\delta R_{\mu\nu}d\tau + \int 2{}^*\mathfrak{R}^{\mu\nu}\delta_{\mu\nu}P d\tau.$$

As before

$$\delta P_{\mu\nu} = \delta S^{\alpha}_{\mu\nu|\alpha} - \delta S^{\alpha}_{\mu\nu|\nu} + S^{\alpha}_{\beta\nu}\delta S^{\beta}_{\mu\alpha} + S^{\beta}_{\mu\alpha}\delta S^{\alpha}_{\beta\nu} - S^{\beta}_{\alpha\beta}\delta S^{\alpha}_{\mu\nu} - S^{\alpha}_{\mu\nu}S^{\beta}_{\alpha\beta}.$$

but it is more convenient now to use the identity

$$\delta S^{\alpha}_{\mu\nu|\alpha} = (\delta S^{\alpha}_{\mu\nu})_{|\alpha} + S^{\beta}_{\mu\nu}\delta\begin{Bmatrix}\alpha\\ \beta\alpha\end{Bmatrix} - S^{\alpha}_{\beta\nu}\delta\begin{Bmatrix}\beta\\ \mu\alpha\end{Bmatrix} - S^{\alpha}_{\mu\beta}\delta\begin{Bmatrix}\beta\\ \nu\alpha\end{Bmatrix}.$$

Only the first term of this identity contributes anything to the S-variation. Integrating by parts where appropriate and rejecting as usual all triple integrals, we find that

$$\delta\int\mathscr{H}_2\,dr = \int({}^*G_{\mu\nu}\delta\mathfrak{g}^{\mu\nu} + \mathfrak{N}^{\mu\nu}_{\alpha}\delta S^{\mu\nu}_{\alpha})d\tau.$$

where

$$\mathfrak{N}^{\mu\nu}_{\alpha} = -2{}^*\mathfrak{R}^{\mu\nu}{}_{|\beta} + {}^*\mathfrak{R}^{\mu\sigma}{}_{|\sigma}\delta^{\nu}_{\alpha} + {}^*\mathfrak{R}^{\nu\sigma}{}_{|\sigma}\delta^{\mu}_{\alpha} + 2{}^*\mathfrak{R}^{\mu\nu} + 2{}^*\mathfrak{R}^{\nu\sigma}S^{\mu}_{\alpha\sigma} - 2{}^*\mathfrak{R}^{\mu\nu}S^{\beta}_{\alpha\beta}$$
$$- {}^*\mathfrak{R}^{\varrho\sigma}S^{\mu}_{\sigma\sigma}\delta^{\nu}_{\alpha} - {}^*\mathfrak{R}^{\varrho\sigma}S^{\nu}_{\varrho\sigma}\delta^{\mu}_{\alpha}$$

and ${}^*G_{\mu\nu}$ remains to be found.

Contracting the equations

$$\mathfrak{N}^{\mu\nu}_{\alpha} = 0.$$

over ν and α, we obtain

$$\mathfrak{R}^{*\mu\sigma}{}_{|\sigma} = {}^*\mathfrak{R}^{\varrho\sigma} S^{\mu}_{\varrho\sigma} - \frac{5}{3}\mathfrak{J}^{\mu}. \tag{28}$$

where, as before,

$$\mathfrak{J}_{\mu} = {}^*\mathfrak{R}^{\mu\nu}{}_{,\nu}.$$

It follows from (27) that

$${}^*\mathfrak{R}^{\mu\nu}{}_{|\alpha} = 2({}^*\underline{\mathfrak{R}}^{\mu\sigma} S^{\nu}_{\alpha\tau} + {}^*\underline{\mathfrak{R}}^{\nu\sigma} S^{\mu}_{\alpha\sigma}) - 4\varphi_{\alpha} {}^*\underline{\mathfrak{R}}^{\mu\nu} - \frac{5}{3}(\mathfrak{J}^{\mu}\delta^{\nu}_{\alpha} + \mathfrak{J}^{\nu}\delta^{\mu}_{\alpha}) \tag{29}$$

If we now express the variation in $R_{\mu\nu}$ in terms of $\left\{ {\lambda \atop \mu\nu} \right\}$ with the help of Palatini's formula already employed in Section 2, integrate by parts, and add the contributions from $\delta S^{\alpha}_{\mu\nu|\nu}$ and $\delta S^{\alpha}_{\mu\alpha|\nu}$, we find that all these terms cancel out. Hence

$$\delta \int \mathscr{H}_2 d\tau = \int (\sqrt{-g} H_{\mu\nu} \delta g^{\mu\nu} + \mathfrak{N}^{\alpha}_{\mu\nu} \delta S^{\alpha}_{\mu\nu}) d\tau,$$

where

$$H_{\mu\nu} = g^{\alpha\beta}({}^*R_{\mu\alpha} {}^*R_{\nu\beta} + {}^*R_{\alpha\mu} {}^*R_{\beta\nu}) - \frac{1}{2} H_2 g^{\mu\nu}.$$

The second set of the field equations is, therefore,

$$H_{\mu\nu} = 0. \tag{30}$$

Recalling that

$${}^*R_{\mu\nu} = {}^*\underline{R_{\mu\nu}} + f_{\mu\nu},$$

these can be written in the form

$${}^*R^{\alpha}_{\mu} {}^*R^{\nu}_{\alpha} + f^{\alpha}_{\mu} f^{\nu}_{\alpha} = \frac{1}{4}\delta^{\nu}_{\mu} H_2, \tag{31}$$

where

$$\underline{{}^*\boldsymbol{R}^{\mu}_{\alpha}} = g^{\alpha\beta} {}^*\underline{R_{\mu\beta}}.$$

Since

$$H_2 = {}^*R_{\mu\nu} {}^*R^{\mu\nu},$$

the field equations (31) remind us of one set of Rainich's algebraic relations:

$$R^{\alpha}_{\mu} R^{\nu}_{\mu} = \frac{1}{4}\delta^{\nu}_{\mu} R_{\alpha\beta} R^{\alpha\beta}.$$

In fact, we can further rewrite (31) to read

$$*\underline{R}^{\alpha}_{\mu}*\underline{R}^{\nu}_{\alpha}+E^{\nu}_{\mu} = \frac{1}{4}\,\delta^{\nu}_{\mu}*R_{\underline{\mu\nu}}*R^{\underline{\mu\nu}}. \tag{32}$$

This form shows explicitly the relation between the electromagnetic energy-stress-momentum tensor E^{ν}_{μ} and the symmetric part of the generalised Ricci tensor $*R_{\underline{\mu\nu}}$.

4. *Discussion*

We have seen that the choice of $S^{\lambda}_{\mu\nu}$ instead of $S^{\sigma}_{\mu\sigma}$ as one of the variables with respect to which action invariant is supposed to be stationary (the other variable is, of course, $\mathfrak{g}^{\mu\nu}$) reduces Weyl's field equations virtually to a form familiar from Rainich's theory. One of the strongest criticisms of the Already Unified Field Theory is that it does not geometrise electromagnetic field in the same sense as gravitation is geometrised in General Relativity. Since equations (27) contain the electromagnetic tensor $E_{\varrho\sigma}$ it may seem reasonable to assume that they will also have written into them a mechanical say $M_{\varrho\sigma}$. Indeed, (27) are equivalent to the generl relativistic equations

$$R_{\varrho\tau}-\frac{1}{2}\,g_{\varrho\sigma}R = -K(M_{\varrho\sigma}-E_{\varrho\sigma}), \tag{33}$$

if

$$K = \beta/*R,$$

and

$$\frac{1}{4}\,Rg_{\varrho\sigma} = KM_{\varrho\sigma}-(S^{\alpha}_{\mu\beta}S^{\beta}_{\alpha\nu}-S^{\alpha}_{\mu\nu}S^{\beta}_{\alpha\beta})\left(S^{\mu}_{\varrho}S^{\nu}_{\sigma}-\frac{1}{4}\,g^{\mu\nu}g_{\varrho\sigma}\right), \tag{34}$$

with

$$R = KM,$$

as required. In this case, equation (33) must be recognised as a definition of the mechanical energy-momentum in terms of S. When $S^{\lambda}_{\mu\nu} = 0$, the distinction between the present theory and General Relativity disappears. The tensor $E_{\mu\nu}$ is then also zero and $*R_{\mu\nu}$ becomes the Ricci tensor $R_{\mu\nu}$ constructed from the ordinary Christoffel brackets. However, the equations

$$R^{\alpha}_{\mu}R^{\nu}_{\ \alpha} = \frac{1}{4}\,\delta^{\nu}_{\mu}R_{\alpha\beta}R^{\alpha\beta},$$

of Rainich, to which (32) would reduce, are a consequence of the algebraic form of $E_{\mu\nu}$ when the field equations are

$$R_{\mu\nu} = KE_{\mu\nu}. \tag{35}$$

Since we obtain (35) from (34) and (32) when $M_{\varrho\sigma} = 0$, it follows that electromagnetic quantities are more elaborately represented in our version of Weyls' theory than they were in Rainich's. The quantity $S^{\lambda}_{\mu\nu}$ describes then really a physical "thing" carrying both electromagnetic and inertial characteristics. It is clearly possible for $f_{\mu\nu}$, and therefore also $E_{\mu\nu}$, to vanish, without $S^{\lambda}_{\mu\nu}$ being zero. In the first version of the theory, however, a purely gravitational field requires that $\beta = 0$ so that β can be only locally constant. Our theory can be regarded, therefore, as complementing the *Already Unified Field.*

But in what sense do we require a non-Riemannian geometry to introduce electromagnetism into its structure? Weyl maintained that in his geometry all non-infinitesimal aspects have been eliminated: rejection of the axiom that parallel transfer of a vector length is integrable led directly to a fusion of gravitation and electromagnetism. The integrability conditions needed in General Relativity to define the reduction of a Riemann space to the flat, gravitationless case, were removed. We have refrained from introducing the notion of length transfer. Matter, whether merely ponderable or carrying an electric charge as well, is represented as a quantitative difference in the definition of parallelism. Since $\Gamma^{\lambda}_{\mu\nu}$ and $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$ are both affine connections, either $A^{\mu} - \Gamma^{\mu}_{\varrho\sigma} A^{\varrho} dx^{\sigma}$ or $A^{\mu} - \left\{ \begin{matrix} \mu \\ \varrho\sigma \end{matrix} \right\} A^{\varrho} dx^{\sigma}$ may be regarded as parallel to A^{μ} at an infinitesimally distant point. The choice between them is *a priori* free and depends on the point of view adopted by a relativistic observer. In other words, we represent matter as a "tilting" of "parallel" vectors. A purely gravitational field is still given by the equations

$$R_{\mu\nu} = 0,$$

and, therefore, preserves, in a sence, a unique character. It must be distinguished from the special relativistic case of an empty space, in which not only $S^{\lambda}_{\mu\nu} = 0$, but also

$$g_{\mu\nu} = \eta_{\mu\nu},$$

the metric tensor of Minkowski.

REFERENCES

[1] H. Weyl, *Sitz. Preus. Akad. Wissenschaft.* 1918 and *Space Time Matter*, London 1922.
[2] A. S. Eddington, *Mathematical Theory of Relativity*, Cambridge 1924.
[3] G. Szekeresz, Private communication.
[4] G. Y. Rainich, *Trabs. Amer. Math. Soc.*, **27**, 106 (1925) and *Mathematics of Relativity*, New York 1950.
[5] A. Einstein, e. g. *Meaning of Relativity*, App. II, London 1954.
[6] E. Schrödinger, *Proc. R. I. A.*, **51**(A); **52**(A), (1948).
[7] A. Einstein, *Sitzungsberichte*, Berlin 1923.
[8] A. H. Klotz, *J. Australian Math. Soc.*, **7**, 613 (1967).
[9] P. A. M. Dirac, *Proc. Roy. Soc.*, **257**, 32 (1960).