Vol. A37 (1970) *ACTA PHYSICA POLONICA* Fasc. 5

PHASE SPACE VOLUME ELEMENT IN INVARIANT VARIABLES

BY J. KŁOSIŃSKI

Physics Department, University of Łódź*

(Received September 17, 1969 and in final form received November 13, 1969)

The paper gives explicit formulas for phase space volume element expressed in Lorentz-invariant variables frequently used in investigations of high energy interactions.

1. *Introduction*

In the experimental investigations and the theoretical description of high energy interactions Lorentz-invariant quantities such as invariant mass squared (IMS) and four-momentum transfer squared (MTS) are used. Often, the amplitude of a process when expressed in these variables posseses a simple form and physical meaning. But in order to obtain (having the amplitude) any distribution or cross-section it is necessary to have an expression for the phase space volume element (PSVE) for the given process.

The aim of this paper is to give explicit formulas for the n-particle PSVE in terms of IMS and MTS variables. N. Byers and C. N. Yang [1] received general expressions for PSVE in terms of scalar products of the four-momenta of the particles in the final state, which are linearly connected with IMS variables. Partially integrated PSVE for study of the IMS distributions in the four and the five particle final states were given by P. Nyborg *et al.* [2]. Results of Ref. [1] and Ref. [2] can be easily reformulated with MTS and IMS as independent variables in reaction processes.

The results received are based on the following property of an N-particle, relativistically invariant, phase space volume [1]:

kinematically allowed regions in $4N$-dimensional phase space, for reaction

$$1+2 \to 3+4+ \dots +N \tag{1}$$

and for decays

$$1 \to \overline{2}+3+4+ \dots +N \tag{2}$$

or

$$\overline{3} \to \overline{1}+\overline{2}+4+ \dots +N \tag{3}$$

* Address: Zakład Fizyki Doświadczalnej Uniwersytetu Łódzkiego, Łódź, Narutowicza 68, Polska.

are different subregions of the same N-particle phase space, which under conditions

$$p_1+p_2=p_3+p_4+\dots+p_N; \quad p_i^2=m_i^2 \tag{4}$$

($p_{\bar{k}}=-p_k$, p_j — four-momentum of j-th particle) reduces to a $(3N-10)$-dimensional restricted phase space.

The subregions are received by setting physical limits on the chosen variables. For example, for reaction (1):

$$M_{34}^2=(p_3+p_4)^2 \geqslant (m_3+m_4)^2, \qquad M_{13}^2=(p_1-p_3)^2 \leqslant (m_1-m_3)^2. \tag{5}$$

Thus, having expressions for PSVE for the decay processes expressed in terms of IMS variables we can easily receive formulas for PSVE for the reaction processes in terms of IMS and MTS variables. For this purpose it is necessary to choose the proper correspondence between particles in the decay processes and the reaction process (or, what is the same, between IMS in the decay and IMS and MTS in the reaction), and set limits of the type (5).

In section II we write the general expression for the PSVE for the decay processes in terms of IMS variables. In section III we give explicit formulas for the PSVE for the reaction processes in IMS and MTS variables frequently used in mutiperipheral models.

2. *General formula for the PSVE for the $(N-1)$-particle decay*

As we have stated in the Introduction, for studying the PSVE for the $(N-2)$-particle final state in the reaction process we must have an expression for the PSVE for the $(N-1)$-particle decay. In both cases this is an N-particle problem with the $(3N-10)$-dimensional restricted phase space.

Let us consider the decay process of particle "0" with the mass m_0 on the $(N-1)$-particles.

$$0 \to 1+2+3+\dots+(N-1). \tag{6}$$

For process (6) PSVE in IMS variables reads [1], [2]:

$$d^{3N-10}\tau_{N,\text{decay}}=\frac{\pi^2|A|^2}{2^{N-1}m_0^2[(-\Delta_1)(-\Delta_2)\dots(-\Delta_{N-4})]^{1/2}}\prod_{i=1}^{3N-10}d\mu_i^2 \tag{7}$$

where the meaning of the symbols is as follows:

$|A|^2$ — S-matrix elements squared and summed with proper weights, over internal quantum numbers,

$\Delta_i's$ — determinants of 4×4 matrices formed out of scalar products of four momenta of particles involved [1],

μ_i^2 — $3N-10$ two particle, three particle, etc. independent IMS's which can be formed for subsystems of particles in final state (6).

Byers and Yang determinants Δ_i, when expressed by IMS's properly chosen, are quadratic functions of these variables. For our selection of IMS's each Δ_i has the form (below in (8) we abandon index i):

$$\begin{aligned}
\Delta = {} & [M^2Q^2+N^2P^2+M^2R^2+N^2R^2+P^2Q^2]- \\
& -2[M^2QR+N^2PR+MNR^2+MPQ^2+NP^2Q]+ \\
& +2[MNPQ+MNPR+MNQR+MPQR+NPQR]- \\
& -2[M^2Qm+N^2Pn+M^2Rm+N^2Rn+MQ^2q+NP^2p+MR^2r+NR^2r+ \\
& +P^2Qq+PQ^2p]- \\
& -2[MNPm+MNQn+MPRp+NQRq+PQRr]+ \\
& +2[MNR(m+n-2r)+MPQ(m+p-2q)+NQP(n+q-2p)+ \\
& +QRM(q+r-2m)+PRN(p+r-2n)]+ \\
& +QRM(q+r-2m)+PRN(p+r-2n)]+ \\
& +[M^2m^2+N^2n^2+P^2p^2+Q^2q^2+R^2r^2]+ \\
& +2[MNmn+MPmp+NQnq+PRpr+QRqr]+ \\
& +2[MQ(mq+mn+qn+mp+qr-pr)+ \\
& +NP(np+nm+pm+pr+nq-qr)+ \\
& +MR(mr+mp+rp+mn+rq-nq)+ \\
& +NR(nr+nq+rq+nm+rp-mp)+ \\
& +PQ(pq+pr+qr+pm+qn-mn)]- \\
& -2[Mm(mp+mn+qr+2np-pr-nq)+ \\
& +Nn(nm+nq+pr-pm-qr\pm 2mq)+ \\
& +Pp(pn+pr+nq-mn-qr+2mr)+ \\
& +Qq(qn+qr+mp-mn-pr+2nr)+ \\
& +Rr(rp+rq+mn-mp-nq+2pq)]+ \\
& +[m^2n^2+m^2p^2+n^2q^2+p^2r^2+q^2r^2]- \\
& -2[m^2np+mn^2q+mp^2r+nq^2r+pqr^2]+ \\
& +2[mnpq+mnpr+mnqr+mpqr+npqr]
\end{aligned} \tag{8}$$

with the following assignemnt of IMS's μ_i^2 to Δ_i's:

$$\begin{aligned}
M_i &= M^2_{i+2,\ldots,N-1}, & N_i &= M^2_{1,2,\ldots,i+1}, \\
P_i &= M^2_{i+1,\ldots,N-1}, & Q_i &= M^2_{1,2,\ldots,i+2}, \\
R_i &= M^2_{i+1,i+2}, & m_i &= M^2_{1,2,\ldots,i}, \\
n_i &= M^2_{i+3,\ldots,N-1}, & p_i &= m^2_{i+2}, \\
q_i &= m^2_{i+1}, & r_i &= m_0^2
\end{aligned}$$

where, by definition,

$$M^2_{i,j,\dots,l} = (p_i+p_j+ \dots +p_l)^2.$$

From a detailed inspection of (7) and (8) it is seen that the denominator in (7) as a whole is a polynomial od second order only in variables P_1, Q_{N-4} and all R_i's. In all other variables it is a polynomial of higher degree. Another property possessed by each Δ_i is a symmetry property [2]. Namely, they are symmetric under simultaneous exchange:

1. $M \leftrightarrow N$; $P \leftrightarrow Q$; $R, r \leftrightarrow R, r$; $m \leftrightarrow n$; $p \leftrightarrow q$ and/or

2. $N \leftrightarrow P$; $Q \leftrightarrow R$; $M, m \leftrightarrow M, m$; $p \leftrightarrow n$; $q \leftrightarrow r$.

Allowed regions in $(3N-10)$-dimensional phase space are given by inequalities $\Delta_i < 0$, $i = 1, \dots, N-4$.

The partial PSVE's are received after integration over some variables. The limits of integration are received as the roots of equation $\Delta_i(\mu_i^2) = 0$, if the integrated variable is any one of the R_i's, P_1 or Q_{N-4}. In this case the Δ_i's have the form $a_i x+2b_i x+c_i$ $(x = \mu_i^2)$. In all other cases the lower limit of integration is received as the largest value of all smaller roots and the upper limit of integration is the smallest value of all larger roots of all quadratic equations (8) in which the integrated variable appears. Equality of the limits of the integration (lower and upper) gives the allowed regions in phase space for the remaining variables present in the given Δ_i.

Partially integrated phase space, when $|A|^2$ do not depend, for example, on one variable, is easily obtained when the denonantor is quadratic in this variable. For example, if it is R_i-th variable then [3]:

$$d^{3N-11}\tau_{N,\text{decay}} = \frac{\pi^3|A|^2}{2^{N-1}m_0^2[\lambda(M_i, N_i, r_i)]^{1/2}} \times$$

$$\times \frac{\prod_{i=1}^{3N-11} d\mu_i^2}{[(-\Delta_i) \dots (-\Delta_{i-1})(-\Delta_{i+1}) \dots (-\Delta_{N-4})]^{1/2}} \tag{9}$$

with the following boundaries on the remaining variables in the i-th determinant:

$$G(P_i, M_i, N_i, m_i, r_i, q_i) = 0; \qquad G(Q_i, M_i, N_i, n_i, r_i, p_i) = 0$$

where:

$$\lambda(a, b, c) = a^2+b^2+c^2-2ab-2bc-2ac$$

$$G(x, y, z, a, b, c) = x^2y+y^2x+z^2a+a^2z+b^2c+c^2b+$$

$$+xzc+xab+yzb+yac-xy(z+a+b+c)-$$

$$-za(x+y+b+c)-bc(x+y+z+a). \tag{10}$$

It should be emphasized (what is commonly known) that also in this method, analitical expressions for partly integrated phase space can be received only to certain degree, after which numerical calculations are necessary.

3. *General expression for PSVE for (N−2)-particle final state in reaction process*

Having PSVE (7) for process (6) we can now, by virtue of the property of phase space cited in the Introduction, write an analogous expression for the process (1). For this aim it is necessary to choose the correspondence between particles in the decay process (6) and the reaction process (1). By "choice of the correspondence" we mean the assignment to each particle in process (6) a particle from reaction (1). Then, variables $M^2_{i,\ldots,k}$ in formula (7) go over into the IMS and MTS variables. Out of many possible correspondences we quote here one which gives variables useful for the study of multiperipheral models. Namely, this correspondence is the following:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \ldots N-1 \\ 2 & \bar{1} & 3 & 4 & 5 \ldots N \end{pmatrix} \begin{matrix} \text{— particles in process (6)} \\ \text{— particles in process (1)} \end{matrix} \tag{11}$$

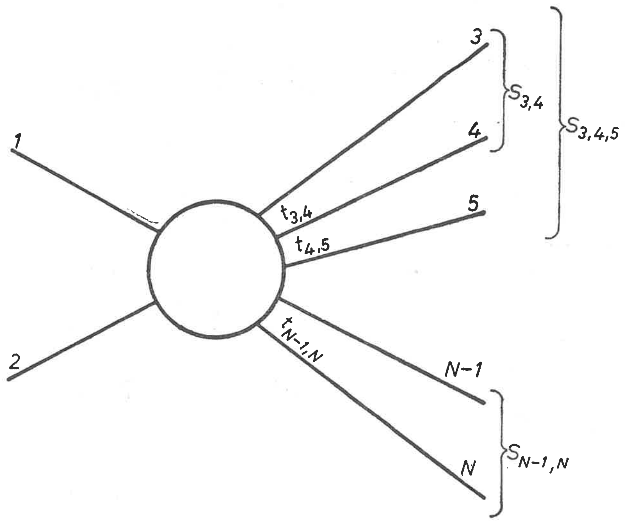

Fig. 1. Parametrization of phase space for correspondence (11) in invariant variables used in multiperipheral models

Then the received set of variables μ_i^2 is as follows (see Fig. 1) — set of two-particle IMS:

$$M^2_{23} = s_{34},\ M^2_{34} = s_{45},\ \ldots,\ s_{N-1,N},$$

— set of momentum transfer MTS $-t_{ij}$ between groups of particles:

$$M^2_{12} = t_{13} \equiv t_{34} \equiv (p_1 - p_3)^2; \quad M^2_{123} = t_{134} \equiv t_{45}; \ldots$$

— set of many-particle IMS:

$$s_{N-2,N-1,N};\ s_{N-3,N-2,N-1,N};\ \ldots$$

— total energy (in CMS) squared

$$s_{3,4,\ldots,N} = s. \tag{12}$$

The i-th Byers and Yang determinant is then

$$M_i = s_{i+3,\ldots,N}; \qquad N_i = t_{i+1,i+2}; \qquad P_i = s_{i+2\,i+3,\ldots,N};$$
$$Q_i = t_{i+3,i+4}; \qquad R_i = s_{i+2,i+3};$$
$$m_i = t_{i+1,i+2}; \qquad m_1 = t_{23} \equiv m_1^2;$$
$$n_i = s_{i+4,i+5,\ldots,N}; \quad n_{N-4} \equiv m_N^2;$$
$$p_i = m_{i+3}^2; \qquad q_i = m_{i+2}^2; \qquad r_i = m_2^2. \tag{13}$$

The above choice is equivalent to some other ones due to symmetry properties of the Byers and Yang determinant.

The production PSVE can be obtained from the decay PSVE by proper normalization [1], *e.g.*,

$$d^{3N-10}\tau_{N,\text{prod.}} = \frac{d^{3N-10}\tau_{N,\text{decay}}}{4\pi(2m_2)^{-1}[\lambda(s, m_1^2, m_2^2)]^{1/2}} \tag{14}$$

and the differential cross-section (for correspondence (11)) is

$$d^{n-4}\sigma_n = \frac{\pi|A|^2 \prod_{i=1}^{3n-4} d\mu_i^2}{2^n(2\pi)^{3n-4}\lambda(s, m_1^2, m_2^2)[(-\Delta_1)(-\Delta_2)\ldots(-\Delta_{N-4})]^{1/2}} \tag{15}$$

where $n = N-2$ and Δ_i's are Byers and Yang determinants (8) expressed by μ_i^2-variables (12), (13).

Partially integrated phase space or the differential cross-section can be obtained by integration by means of a procedure analogous to that used in Sec. II.

When the matrix element squared does not depend on any single variable (statistical process), then by step by step integration we arrive at the well-known expression for the total cross-section [4] for production of n particles:

$$\sigma_n(s) = \frac{\pi^{n-1}}{2^n(2\pi)^{3n-4}\lambda(s, m_1^2, m_2^2)} I_n(s, m_3, m_4, \ldots, m_N) \tag{16}$$

with the following recurrence formula:

$$I_n(s, m_3, m_4, \ldots, m_N) = \frac{1}{s} \int_{(m_3+m_4+\ldots+m_{N-1})^2}^{(\sqrt{s}-m_N)^2} [\lambda(s, x, m_N^2)]^{1/2} I_{n-1}(x, m_3, \ldots, m_{N-1})dx$$

$$I_2(x, m_3, m_4) = \frac{1}{x}[\lambda(x, m_3^2, m_4^2)]^{1/2}. \tag{17}$$

The asymptotical cross-section (for $s \to \infty$) can be also very easily received [5]. The general formula for differential cross-section (15) is also very convenient for analysis [6].

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