GAUGE INVARIANCE OF QUANTUM ELECTRODYNAMICS

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Is is shown using perturbation theory that the transition amplitudes in quantum electrodynamics are gauge invariant; taking into account the gauge dependence of the renormalization constant Z_2 for the electron field. A class of gauges which are generalization of the Coulomb gauge are considered.

Introduction

It has been pointed out by Białynicki-Birula [1] that the theorem on gauge invariance of transition amplitudes in quantum electrodynamics, as given by Feynman [2] and used by many others [1], is incorrect. The aim of the present work is to prove the invariance of the S matrix elements using the perturbation theory under the gauge transformations and taking into account the gauge dependence of the renormalization constant Z_2 for the electron field. We shall make use of the Feynman diagram technique. In perturbation theory, different gauges correspond to different free photon propagators. The Fourier transform of the electron propagator, in the lowest order of the perturbation theory, is the same for all gauges and is given by

$$S_F(p) = \frac{1}{\hat{p}-m}.$$

We shall consider a class of gauges which are a generalization of the Coulomb gauge. For this class of gauge transformations the photon propagator $D_{\mu\nu}(k)$ becomes

$$\tilde{D}_{\mu\nu} = D_{\mu\nu}(k) + k_\mu D_{\nu\lambda} f^\lambda + D_{\mu\lambda} f^\lambda k_\nu + k_\mu k_\nu D_{\varrho\lambda} f^\varrho f^\lambda \tag{1}$$

where $f_\mu(k)$ is a real and odd function of k_μ depending on the arbitrary number of the external vectors. We shall assume that the S matrix element for a given process, in the n-th order of perturbation theory, is calculated by taking the mass shell limit of the expression obtained from Feynman diagrams multiplied by $(p\text{-}m)\frac{1}{\sqrt{z_2}}$ for each electron and $\frac{q^2}{\sqrt{z_3}}$ for each pho-

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ton, and by inserting the free particle wave functions for each external line. Since in the expression for the S matrix element there appear renormalization constants, we must give a method for their calculation for the class of gauge transformations considered.

Generalization of the renormalization constant

The electron propagator in the relativistic gauges has the spectral representation [3]

$$S'_F(p) = \frac{Z_2(\hat{p}+m)}{p^2-m^2+i\varepsilon} + \int_{m_2}^{\infty} dM^2 \frac{\hat{p}\varrho_1(M^2)+\varrho_2(M^2)}{p^2-M^2+i\varepsilon} \tag{2}$$

where Z_2 is the gauge dependent renormalization constant for the electron field. As $\hat{p} \to m$, we find that

$$S'_F \to \frac{Z_2}{\hat{p}-m} \tag{3}$$

Z_2 is found from the expression for S'_F in the n-th order of the perturbation theory

$$Z_2(\hat{p}+m) = \lim_{p^2\to m^2} (p^2-m^2)\, S'_F(p). \tag{4}$$

In a general case we encounter certain difficulties, *viz.*, in the radiation guage [4, 5] we have a spectral representation of the form

$$S'_F(p^2,\bar{p}^2) = \int_0^{\infty} \frac{A_1(\bar{p}^2, M^2)\gamma^0 p^0 + A^2(\bar{p}^2,M^2)\,\overline{\gamma p} + C(\bar{p}^2,M^2)}{p^2-M^2+i\varepsilon}\, dM^2, \quad \bar{p}^2 = p_1^2+p_2^2+p_3^2. \tag{5}$$

There are three spectral functions instead of the customary two and in the neighbourhood of $p^2 = m^2$ the usual form of the Green function is not applicable.

Hagen [5] assumed an expression sufficiently general in this domain,

$$S'_F \sim e^{W(\bar{p}^2)\overline{\gamma p}}\, \frac{Z_2(\bar{p}^2)}{\hat{p}-m}\, e^{W(\bar{p}^2)\overline{\gamma p}}, \quad W^*(\bar{p}^2) = W(\bar{p}^2), \tag{6}$$

and found $Z_2(\bar{p}^2)$ and $W(\bar{p}^2)$ in the 2-nd order of the perturbation theory. The expression (6) corresponds to

$$S'_F \sim \frac{A_1\gamma^0 p^0 + A_2\,\overline{\gamma p} + A_3 m}{p^2-m^2} \tag{7}$$

and

$$S'_F \sim \frac{N\,(\hat{p}+m)\,N}{p^2-m^2} \tag{8}$$

where

$$N = \sqrt{Z_2(\bar{p}^2)}\, e^{W(\bar{p}^2)\overline{\gamma p}} = B_1(\bar{p}^2) + B_2(\bar{p}^2)\,\overline{\gamma p} \tag{9}$$

near the pole.

Hence we see that N corresponds to $\sqrt{Z_2}$ for relativistic gauges. Even though N is not a constant but a matrix function of $\bar{p}$, we shall continue to refer to it as renormalization constant. For gauges from the class under consideration we have

$$D_{\mu\nu} = -\frac{1}{k^2}(g_{\mu\nu}+f_\mu k_\nu+k_\mu f_\nu+k_\mu k_\nu f^2) \tag{10}$$

where f_μ may depend on the arbitrary number of external vectors (the rotational invariance is broken).

If we assume the C, P and T invariance, the general expression for the electron propagator near the pole has the form

$$S'_F \sim \frac{A_1\,\gamma^0p^0+A_2\gamma^1p^1+A_3\,\gamma^2p^2+A_4\,\gamma^2p^3+A_5\gamma^0\gamma^1\gamma^2\gamma^3p^0p^1p^2p^3}{p^2-m^2} \tag{11}$$

where A_i $(i = 1 \dots 5)$ are the real functions of p_l^2 and p_lp_k $l, k = 1, 2, 3$.

We may then write

$$S'_F \sim \frac{\bar{N}(\hat{p}+m)\,N}{p^2-m^2} \qquad \bar{N} = \gamma^0N^+\gamma^0 \tag{12}$$

and

$$N = B_1+B_2\gamma^1p^1+B_3\gamma^2p^2+B_4\gamma^3p^3+B_5\gamma^1\gamma^2p^1p^2+B_6\gamma^1\gamma^3p^1p^3+$$
$$+B_7\gamma^2\gamma^3p^2p^3+B\gamma^1\gamma^2\gamma^3p^1p^2p^3 \tag{13}$$

B_α $\alpha = 1, 2, \dots 8$ are real functions of p_l^2 and p_lp_k $l, k = 1, 2, 3$. In general, however, $N \neq \bar{N}$. From the form of the electron propagator near the pole it is not possible to determine N uniquely (more unknown quantities than equations). In the radiation gauge we have only one external vector n_μ $n_\mu n^\mu = 1$, and the number of B_α's reduces so that $N = \bar{N}$ and N is determined uniquely.

There is also some freedom in the determination of N. If we introduce $\bar{N}' = \bar{N}\frac{\hat{p}}{m}$ and $N' = \frac{\hat{p}}{m}N$ we obtain

$$\bar{N}'(\hat{p}+m)\,N' = N(\hat{p}+m)\,N \text{ and } \bar{N}' \neq N'.$$

We shall assume that γ^0p^0 is eliminated from the expression for N by substituting $\gamma^0p^0 = \overline{\gamma p}+m$.

Change of the expression obtained from Feynman graphs under gauge transformations

We consider infinitesimal transformations, in view of the group property of the gauge transformations. For the photon propagators in the lowest order of the perturbation theory we obtain for infinitesimally different gauges

$$\tilde{D}_{\mu\nu} = D_{\mu\nu}+k_\mu D_{\nu\lambda}\delta f^\lambda+k_\nu D_{\mu\lambda}\delta f^\lambda. \tag{14}$$

Let us consider all the Feynman diagrams with m external lines giving contribution to the S matrix element in the n-th order of the perturbation theory.

We want to find the change in the corresponding expression due to the change of the photon propagator. This change corresponds to the replacement of the propagator of every internal photon line by

$$D_{\mu\nu}+k_\mu D_{\nu\lambda}\delta f^\lambda+k_\nu D_{\mu\lambda}\delta f^\lambda.$$

and taking into account the linear part in δf^λ only.

In other words, we take the sum of the diagrams without changing any of the electron lines nor any, save one, photon lines, and substitute for this one line the extra part of the photon propagator

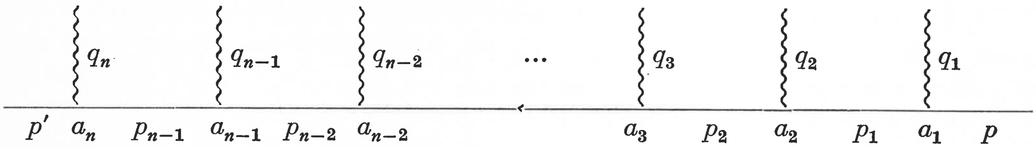
$$k_\mu D_{\nu\lambda}\delta f^\lambda+k_\nu D_{\mu\lambda}\delta f^\lambda.$$

We divide the contribution from this line into two independent parts and sum over two possible choices of endpoints,

$$\underset{\sim\sim\sim}{k_\mu D_{\nu\lambda}\delta f^\lambda}+\underset{\sim\sim\sim}{k_\nu D_{\mu\lambda}\delta f^\lambda} = \underset{\sim\sim\sim\sim\sim\sim}{k_\mu D_{\nu\lambda}\delta f^\lambda+k_\nu D_{\mu\lambda}\delta f^\lambda}.$$

In the next part of the proof we will use a result of Feynman which we shall call a lemma.

Let us consider a fermion line (being a part of the general diagram) to which n photons, real or virtual, have been attached. This graph,



is given by the amplitude

$$F(p',p)=\frac{1}{\hat{p}'-m}\,\hat{a}_n\,\frac{1}{\hat{p}_{n-1}-m}\,\hat{a}_{n-1}\,\frac{1}{\hat{p}_{n-2}-m}\,\hat{a}_{n-2}\dots\hat{a}_3\,\frac{1}{\hat{p}_2-m}\,\hat{a}_2\,\frac{1}{\hat{p}_1-m}\,\hat{a}_1\,\frac{1}{\hat{p}-m}. \tag{15}$$

If we insert a photon line, real or virtual, which has the momentum $\hat{q}$ and interaction vertex $\hat{q}$, and sum over all possible positions r where the photon can be inserted, we see that the terms cancel pairwise. Only two terms resulting from inserting the photon q on the right-hand side of q_1 and on the left-hand side of q_n escape cancellation, and give the result

$$\begin{aligned}\tilde{F}=&\frac{1}{\hat{p}'-m}\,\hat{a}_n\,\frac{1}{\hat{p}_{n-1}-m}\,\hat{a}_{n-1}\dots\hat{a}_2\,\frac{1}{\hat{p}_1-m}\,\hat{a}_1\,\frac{1}{\hat{p}-m}-\\ &-\frac{1}{\hat{p}'+\hat{q}-m}\,\hat{a}_n\,\frac{1}{\hat{p}_{n-1}+\hat{q}-m}\,\hat{a}_{n-1}\dots\hat{a}_2\,\frac{1}{\hat{p}_1+\hat{q}-m}\,\hat{a}_1\,\frac{1}{\hat{p}+\hat{q}-m}.\end{aligned} \tag{16}$$

If the fermion line is an open line in an S matrix element, p and $p'+q$ are momenta of real external particles and the contribution of the graph to the S matrix is obtained by removing the external propagator legs

$$S\sim\bar{u}(p'+q)\left[\lim_{\substack{p^2\to m^2\\(p'+q)^2\to m^2}}(\hat{p}'+\hat{q}-m)\,\frac{1}{\sqrt{Z_2}}\,\tilde{F}(p',p)\frac{1}{\sqrt{Z_2}}\,(\hat{p}-m)\right]u(p). \tag{17}$$

Feynman in his proof assumed that this contribution is null, but this assumption is not valid. We shall see that when there are radiative corrections to the external electron lines, factors $\dfrac{1}{\hat{p}'+\hat{q}-m}$ and $\dfrac{1}{\hat{p}-m}$ singular on the mass shell occur in the expression for F

making the contribution differ from zero. As it is seen from the expression for the S matrix element we must first perform the multiplication by $\hat{p}-m$ and $\hat{p}'+\hat{q}-m$, and pass on to the mass shell. Subsequently, we multiply by the spinors $u(p)$ and $\bar{u}(p'+q)$ (the use of the equation $(\hat{p}-m)\,u(p) = 0$ means that in the configuration space integration by parts is employed and that adequate account must be taken of the boundary terms). If the fermion line is a closed loop, the contribution to the S matrix element is

$$S \sim \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}[\hat{a}_0 \tilde{F}] = \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}[a_0 (F(p',p) - F(p'+q, p+q))] \tag{18}$$

and vanishes provided the origin of integration in the momentum space may be shifted (we assume that all divergent integrals are regularized in such a way that the origin may be shifted). Let us now consider the sum of the Feynman graphs giving contribution to the S matrix element. If we take a diagram with one endpoint fixed corresponding to the vertex $\gamma^\mu D_{\mu\lambda}\delta f^\lambda = \delta\hat{f}'$ and a second endpoint corresponding to the vertex $\hat{k}$, attached to a certain electron line, the sum encompasses diagrams with all possible positions of the vertex $\hat{k}$; apart from that the diagrams remain unchanged.

Using the procedure described by Feynman's lemma, we can express this class of diagrams by the difference of two diagrams with appropriate momenta but with one vertex less. We shall show that some classes of diagrams give zero contribution when multiplied by $\hat{p}$-m and by the corresponding spinors, and we find an expression for the nonvanishing contribution. It may be shown that for external lines the infinitesimal change of the potential A_μ (proportional to the four momentum) gives no contribution to the observable processes). Feynman's original proof gives here a correct result, *cf.* Bogolubov [6]). We shall consider only internal photon lines and we shall distinguish five cases:

(*a*) the photon propagator has both its endpoints on the same open electron line;

(*b*) the photon propagator has both its endpoints on the closed electron line;

(*c*) the photon line has its endpoints on two different electron lines which are both open;

(*d*) the photon propagator has its endpoints on different electron lines one of which is closed, and

1° vertex $\delta\hat{f}'$ is on the open electron line, $\hat{k}$ on the closed one,

2° vertex $\hat{k}$ is on the open electron line, $\delta\hat{f}'$ on the closed one;

(*e*) the photon propagator has its endpoints on different electron lines both of which are closed.

From the lemma and discussion of the closed electron lines it is seen that the contribution to the S matrix element is zero for the diagrams corresponding to the cases (*b*), (*e*) and (*d*) 1°. Let us assume that the endpoint corresponding to $\hat{k}$ is on the open electron line. From the expression for the S matrix element

$$S \sim \bar{u}(p'+q)\left[\lim_{\substack{p^2\to m^2\\(p'+q)^2\to m^2}} (\hat{p}'+\hat{q}-m)\frac{1}{\sqrt{Z_2}}\left[\frac{1}{\hat{p}'-m}\hat{a}_n\frac{1}{\hat{p}_{n-1}-m}\hat{a}_{n-1}\dots\hat{a}_2\frac{1}{\hat{p}_1-m}\hat{a}_1\frac{1}{\hat{p}-m}-\right.\right.$$

$$\left.\left.-\frac{1}{\hat{p}'+\hat{q}-m}\hat{a}_n\frac{1}{p_{n-1}+\hat{q}-m}\hat{a}_{n-1}\dots\hat{a}_2\frac{1}{\hat{p}_1+\hat{q}-m}\hat{a}_1\frac{1}{\hat{p}+\hat{q}-m}\right]\frac{1}{\sqrt{Z_2}}(\hat{p}-m)\right]u(p) \tag{19}$$

we see that the contribution differs from zero when p_i, p_j exist such that $p_i+q = p$ or $p'+q = p_j$.

Let us consider the first case. We must nave $q = -\sum_{k=1}^{i} q_k$, where q_k are the momenta of the photon lines whose endpoints can be located:

1. on the same electron line (number of vertex smaller or greater than i);
2. on different electron lines, open or closed.

However, not all q are independent, since we can have in $\sum_{k=1}^{i} q_k$ cancellations corresponding to self energy insertions

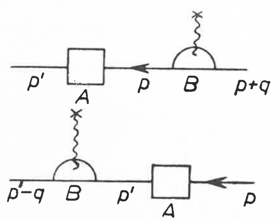
$$(q_l+q_m = 0 \quad l, m < i).$$

It is also possible that a group of some momenta can be expressed (making use of the fact that the sum of momenta incoming to a closed loop is zero) by momenta of the external lines, momenta of the lines corresponding to vertexes with numbers greater than i or momenta of lines with endpoints on different open electron lines. We have $q = -\sum_k q'_k$, where q'_k are external momenta or the momenta which are independent variables of integration.

Because of the independence of q'_k, the equality $q = -\sum_k q'_k$ is possible only in the case when the sum reduces to one term $q = -q'_k$. This means that the sum of i momenta of the corresponding photon lines must be either equal to the momentum of the one photon line or to the sum of the groups of momenta of photon lines whose other endpoints are on the closed electron lines. The discussion of the case $p'+q = p_j$ is analogous. However, the two conditions $p'+q = p_j$ and $p_i+q = p$ cannot be fulfilled at the same time. From the two terms in the expression for the S matrix element only one gives a nonvanishing contribution.

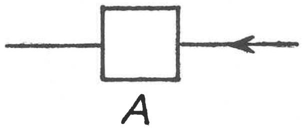
It follows from the discussion that in the case (*c*) the contribution to the S matrix element is zero. From the expression for the S matrix element we can see that for a class of diagrams, when the endpoint of the photon line corresponding to $\delta\hat{f}'$ is attached at such a point of the electron line that on both its sides photon lines with the other endpoints on the open electron lines or external lines occur, the contribution vanishes.

For subclasses of diagrams for the cases (*a*) and (*d*) 2° we obtain results different from zero. This may be presented schematically as follows:

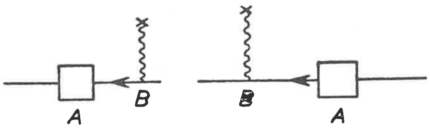


We find what diagrams correspond to part B, and draw several of the lowest order diagrams

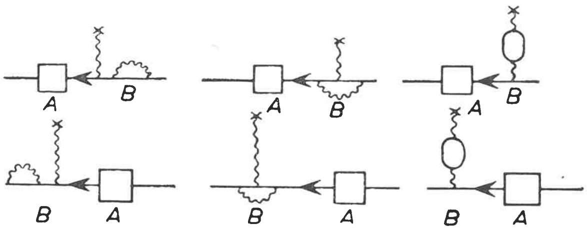
Part B zero order



Part B first order



Part B third order



In the higher orders of the perturbation theory the procedure will be analogous. It is seen from the diagrams that in part B we have corrections to the electron propagator, vertex part, and photon propagator. Taking into account the normalization factors we obtain for part B the expressions for the incoming electron line

$$-ie^2 \int \Gamma^\nu(p, p+k) S'_F(p+k) D'_{\mu\nu} \delta f^\mu \frac{d^4k}{(2\pi)^4} \tag{20}$$

and the outgoing electron line

$$-ie^2 \int \frac{d^4k}{(2\pi)^4} \delta f^\mu D'_{\mu\nu} S'_F(p'+k) \Gamma^\nu(p'+k, p'). \tag{21}$$

Let $F_{\mu_1} \cdots \mu_l (p', p'_1 \cdots p'_k, p, p_1 \cdots p_k, q_1 \cdots q_l)$ denote an expression obtained from Feynman diagrams for the process with m external lines in n-th order of the perturbation theory. The change of this expression under infinitesimal gauge transformation of the photon propagator is given by (contribution from one open electron line with momenta p, p'),

$$S \sim \bar{u}(p')(\hat{p}'-m) \frac{1}{\bar{N}} \prod_{j=1}^{k} \bar{u}(p'_j)(\hat{p}'_j - m) \frac{1}{\bar{N}_j} \delta F_{\mu_1 \ldots \mu_l}(p', p'_1 \ldots p'_k, p, p_1 \ldots p_k, q_1 \ldots q_l) \times$$

$$\times \prod_{i=1}^{k} \frac{1}{N_i} (\hat{p}_i - m) u(p_i) \times$$

$$\times \frac{1}{N}(\hat{p}-m)\,u(p)\prod_{r=1}^{l} q_r^2 \frac{1}{\sqrt{Z_3}} A_{\mu_r} = \bar{u}(p')\left[(\hat{p}'-m)\frac{1}{N}\,\delta F(p',p)\frac{1}{N}(\hat{p}-m)\right]\times$$

$$\times\, u(p) = \bar{u}(p')\left[[\hat{p}'-m)\frac{1}{N}F(p',p)(-ie^2)\int \Gamma^{\nu}(p,p+k)\,S'_F(p+k)\,D'_{\mu\nu}\,\delta f^{\mu}\times\right.$$

$$\times \frac{d^4k}{(2\pi)^4}\frac{1}{N}(\hat{p}-m)]\,u(p)+$$

$$+\,\bar{u}(p')\left[(\hat{p}'-m)\frac{1}{N}(-ie^2)\int\frac{d^4k}{(2\pi)^4}D'_{\mu\nu}\,\delta f^{\mu}\,S'_F(p'+k)\,\Gamma^{\nu}(p'+k,p')\,F(p',p)\times\right.$$

$$\times\frac{1}{N}(\hat{p}-m)]u(p). \tag{22}$$

When we have a diagram with a fixed endpoint corresponding to $\delta\hat{f}'$, we can give a class of diagrams in which the endpoint $\hat{k}$ of the photon line is attached in all possible ways to the electron line considered, the remaining part of the diagram is unchanged. This class, according to the lemma, gives a contribution to the S matrix element which either vanishies or differs from zero, and in the latter case we can ascribe to this class a diagram from the group of diagrams described by the expression

$$F(-ie^2)\int\Gamma^{\nu}(p,p+k)\,S'_F(p+k)\,D'_{\mu\nu}\,\delta f^{\mu}\frac{d^4k}{(2\pi)^4}$$

$$\text{or}\quad -ie^2\int\delta f^{\mu}D'_{\mu\nu}S'_F(p'+k)\,\Gamma^{\nu}(p'+k,p')\frac{d^4k}{(2\pi)^4}F. \tag{23}$$

Part B described by the above expressions must include all corrections to $D'_{\mu\nu}S'_F$ and Γ^{ν}, because we have started from a group encompassing all possible diagrams giving contribution to the S matrix element.

Conversely, to every diagram from the group of diagrams corresponding to the expression (23)

$$F(-ie^2)\int\Gamma^{\nu}(p,p+k)\,S'_F(p+k)\,D'_{\mu\nu}\delta f^{\mu}\frac{d^4k}{(2\pi)^4}\ \text{or}\ -ie^2\int\delta f^{\mu}D'_{\mu\nu}S'_F(p'+k)\Gamma^{\nu}(p'+k,p')\frac{d^4k}{(2\pi)^4}F$$

we may ascribe in a unique way the class of Feynman diagrams from the group of diagrams we started with. If we fix the part B to be of the k-th order in the perturbation theory, part A will include all the diagrams corresponding to the S matrix element of the n–k–1 order. If we were to find a diagram missing in A, this would mean that the starting group of diagrams was incomplete.

Analogously we find an expression for the change of the electron propagator under infinitesimal gauge transformation of the photon propagator

$$\delta S'_F = -ie^2\int S'_F(p)\,\Gamma^{\nu}(p,p+k)\,S'_F(p+k)\,D'_{\mu\nu}\delta f^{\mu}\frac{d^4k}{(2\pi)^4}-$$

$$-ie^2\int S'_F(p+k)\,\Gamma^{\nu}(p+k,p)\,S'_F(p)\,D'_{\mu\nu}\delta f^{\mu}\frac{d^4k}{(2\pi)^4}. \tag{24}$$

Gauge independence of the S matrix element (relativistic gauges)

Let us assume that the photon propagators in the two relativistic gauges have the form

$$D_{\mu\nu} = -\frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \text{ and } \tilde{D}_{\mu\nu} = -\frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + d(k^2)\frac{k_\mu k_\nu}{k^2}\right)$$

$$\delta f^\mu = -\frac{k^\mu}{k^2}\alpha(\lambda)\,\delta\lambda. \tag{25}$$

From the expression for the infinitesimal gauge transformation of the photon propagator

$$\delta D_{\mu\nu} = \left[k_\mu D_{\nu\varrho}\frac{k^\varrho}{k^2} + k_\nu D_{\mu\varrho}\frac{k^\varrho}{k^2}\right]\alpha(\lambda)\,\delta\lambda. \tag{26}$$

Let us further assume

$$D_{\mu\nu}(\lambda) = -\frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + F(\lambda)\frac{k_\mu k_\nu}{k^2}\,d(k^2)\right). \tag{27}$$

We then have

$$\frac{dF}{d\lambda} = 2F(\lambda)\,\alpha(\lambda) \tag{28}$$

and taking the conditions $\alpha(\lambda) = \frac{1}{2\lambda}$

$$\begin{aligned} \lambda = 0 \quad & D_{\mu\nu}(\lambda) = D_{\mu\nu} \\ \lambda = 1 \quad & D_{\mu\nu}(\lambda) = D_{\mu\nu} \end{aligned} \tag{29}$$

we obtain

$$D_{\mu\nu}(x) = -\frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \lambda\frac{k_\mu k_\nu}{k^2}\,d(k^2)\right). \tag{30}$$

The electron propagator in the relativistic gauge has the following form near its pole

$$S'_F \sim \frac{Z_2(\hat{p}+m)}{p^2-m^2} = \frac{Z_2}{\hat{p}-m}. \tag{31}$$

$$(\hat{p}+m)\,Z_2 = \lim_{p^2\to m^2}(p^2-m^2)S'_F(p) \quad \text{or} \quad Z_2 u(p) = [S'_F(\hat{p}-m)]u(p)$$

δZ_2 is obtained from the relation

$$\delta Z_2\,u(p) = [\delta S'_F(\hat{p}-m)]\,u(p) = \left[\left(-ie^2\frac{\delta\lambda}{2\lambda}\int S'_F(p)\Gamma^\nu(p,p+k)\,S'_F(p+k)\,D'_{\mu\nu}\frac{k^\mu}{k^2}\frac{d^4k}{(2\pi)^4} -\right.\right.$$

$$\left.\left. -ie^2\frac{\delta\lambda}{2\lambda}\int S'_F(p+k)\,\Gamma^\nu(p+k,p)\,S'_F(p)\,D'_{\mu\nu}\frac{k^\mu}{k^2}\frac{d^4k}{(2\pi)^4}\right)(\hat{p}-m)\right]u(p) \tag{32}$$

Γ^ν, S'_F and $D'_{\mu\nu}$ are functions of λ. From $D'_{\mu\nu}k^\nu = -\frac{1}{k^2}k_\mu\lambda d(k^2)$ and the generalized.

Ward identity we have

$$\delta Z_2(\lambda)\, u(p) = ie^2 \int \frac{d^4k}{(2\pi)^4}\, Z_2(\lambda)\, u\, \frac{d(k^2)}{k^4}\, \delta\lambda \tag{33}$$

and from the expression for $\delta F(p', p)$, obtained from the perturbation theory, we see that the change of the renormalization constant Z_2 compensates the change under gauge transformation of the expression obtained from the Feyman diagrams describing the process.

After integration we obtain from the equation for $Z_2(\lambda)$ when $\lambda = 1$ the known [7, 8] relation

$$\tilde{Z}_2 = Z_2 \exp ie^2 \int \frac{d(k^2)}{k^4}\, \frac{d^4k}{(2\pi)^4}\,. \tag{34}$$

Gauge independence of the S matrix element (general case)

Suppose the photon propagator in a gauge from the considered class of gauge transformations has the form

$$\tilde{D}_{\mu\nu} = -\frac{1}{k^2}\,(g_{\mu\nu} + k_\mu f_\nu + k_\nu f_\mu + k_\mu k_\nu f^2). \tag{35}$$

The expression for the change of the photon propagator under infinitesimal gauge transformations gives

$$\delta D_{\mu\nu}(\lambda) = (k_\mu D_{\nu\lambda} f^\lambda + k_\nu D_{\mu\lambda} f^\lambda)\,\alpha(\lambda)\,\delta\lambda \qquad \delta f^\mu = f^\mu \alpha(\lambda)\,\delta\lambda. \tag{36}$$

We assume that

$$D_{\mu\nu}(\lambda) = -\frac{1}{k^2}\,(g_{\mu\nu} + (k_\mu f_\nu + k_\nu f_\mu)\, F(\lambda) + k_\mu k_\nu f^2 G(\lambda)) \tag{37}$$

hence,

$$\frac{dF}{d\lambda} = [1 + (kf) F(\lambda)]\,\alpha(\lambda) \qquad \frac{dG}{d\lambda} = 2[F(\lambda) + (kf)\, G(\lambda)]\,\alpha(\lambda) \tag{38}$$

taking $\alpha(\lambda) = \dfrac{1}{1+\lambda(kf)}$ and the conditions

$$D_{\mu\nu}(0) = D_{\mu\nu} \qquad D_{\mu\nu}(1) = \tilde{D}_{\mu\nu} \tag{39}$$

we obtain

$$D_{\mu\nu}(\lambda) = -\frac{1}{k^2}\,(g_{\mu\nu} + (k_\mu f_\nu + k_\nu f_\mu)\lambda + \lambda^2 k_\mu k_\nu f^2). \tag{40}$$

Let us assume that the electron propagator has the following form near the pole:

$$S'_F \sim \frac{\overline{N}(\hat{p} + m) N}{p^2 - m^2} \tag{41}$$

and in an infinitesimally different gauge

$$\tilde{S}'_F \sim \frac{(\bar{N}+\delta\bar{N})(\hat{p}+m)(N+\delta N)}{p^2-m^2}. \tag{42}$$

The difference can be written

$$\delta S'_F = \frac{\bar{N}(\hat{p}+m)\delta N+\delta\bar{N}(\hat{p}+m)N}{p^2-m^2}. \tag{43}$$

From the expression for $\delta S'_F$ obtained from the perturbation theory we have

$$\delta S'_F = -ie^2 \int S'_F(p)\Gamma'^\nu(p,p+k)S'_F(p+k)D'_{\mu\nu}\delta f^\mu \frac{d^4k}{(2\pi)^4} -$$

$$-ie^2 \int S'_F(p+k)\Gamma^\nu(p+k,p)\, S'_F(p)\, D'_{\mu\nu}\delta f^\mu \frac{d^4k}{(2\pi)^4}$$

and near $p^2 = m^2$ we have

$$\delta S'_F = -ie^2\delta\lambda \int \frac{\bar{N}(\hat{p}+m)\, N}{p^2-m^2}\, \Gamma^\nu(p,p+k)\, S'_F(p+k)\, D'_{\mu\nu}f^\mu \frac{1}{1+(kf)\,\lambda}\frac{d^4k}{(2\pi)^4} -$$

$$- ie^2\, \delta\lambda \int S'_F(p+k)\, \Gamma^\nu(p+k,p)\, D'_{\mu\nu}f^\nu \frac{d^4k}{(2\pi)^4}\frac{1}{1+(kf)\lambda}\frac{\bar{N}(\hat{p}+m)\, N}{p^2-m^2}. \tag{44}$$

Let us consider two cases:

1. Renormalization constant N can be uniquely determined in the perturbation theory from the expression for $S'_F(\lambda)$ in the pole. Comparing two expressions for $\delta S'_F$ we obtain

$$\delta N = - ie^2\, \delta\lambda\, N \int \Gamma^\nu(p,p+k)\, S'_F(p+k)\, D'_{\mu\nu}f^\mu \frac{1}{1+(kf)\lambda}\frac{d^4k}{(2\pi)^4} \tag{45}$$

and for the contribution to the S matrix element (from one open electron line) under an infinitesimal gauge transformation of the photon propagator we have

$$\delta \left[\bar{u}(p')\,(\hat{p}'-m)\,\frac{1}{\bar{N}}\,F(p',p)\,\frac{1}{N}\,(\hat{p}-m)\,u(p)\right]$$

$$=\bar{u}(p')\,(\hat{p}'-m)\,\delta\left(\frac{1}{\bar{N}}\right) F(p',p)\,\frac{1}{N}\,(\hat{p}-n)\,u(p) +$$

$$+\,\bar{u}(p')\,(\hat{p}'-m)\,\frac{1}{\bar{N}}\,F(p',p)\,\delta\left(\frac{1}{N}\right)(\hat{p}-m)\,u(p)+\bar{u}(p')\,(\hat{p}'-m)\,\times$$

$$\times\,\frac{1}{\bar{N}}\,\delta F(p',p)\,\frac{1}{N}\,(\hat{p}-m)\,u(p)$$

$$= -\,\bar{u}(p')\,(\hat{p}'-m)\,\frac{1}{\bar{N}}\,\delta\bar{N}\,\frac{1}{\bar{N}}\,F(p',p)\,\frac{1}{N}\,(\hat{p}-m)\,u(p) -$$

$$- \bar{u}(p')(\hat{p}'-m)\frac{1}{\bar{N}}F(p',p)\frac{1}{N}\,\delta N\frac{1}{N}(\hat{p}-m)\,u(p)\,+$$

$$+\,\bar{u}(p')(\hat{p}'-m)\frac{1}{\bar{N}}\,\delta\lambda\,(-ie^2)\int D'_{\mu\nu}f^{\mu}\alpha(\lambda)S'_F(p'+k)\,\Gamma^{\nu}(p'+k,p')\,\times$$

$$\times\frac{d^4k}{(2\pi)^4}F(p',p)\frac{1}{N}(\hat{p}-m)\,u(p)\,+$$

$$+\,\bar{u}(p')(\hat{p}'-m)\frac{1}{\bar{N}}F(p',p)(-ie^2)\,\delta\lambda\int\Gamma^{\nu}(p,p+k)\,S'_F(p+k)\,D'_{\mu\nu}f^{\mu}\alpha(\lambda)\,\times$$

$$\times\frac{d^4k}{(2\pi)^4}\frac{1}{N}(\hat{p}-m)\,u(p)=0. \tag{46}$$

After summation over all open electron lines we obtain gauge independence of the S matrix element.

2. Renormalization constant N cannot be uniquely determined from the expression for $S'_F(\lambda)$ near the pole.

To define unambiguously the change of the renormalization constant under infinitesimal gauge transformation we can use the relation

$$\delta N=-ie^2\delta\lambda N(\lambda)\int\Gamma^{\nu}(p,p+k)S'_F(p+k)D'_{\mu\nu}f^{\mu}\frac{1}{1+(kf)\lambda}\frac{d^4k}{(2\pi)^4}. \tag{47}$$

The electron propagator described near the pole by the expression

$$S'_F\sim\frac{\bar{N}(\hat{p}+m)N}{p^2-m^2} \tag{48}$$

under gauge transformation, with δN as given above, does not change its mode of behaviour. For δN defined in this way we obtain analogically as in case 1 gauge independence of the S matrix element.

The differential equation for $N(\lambda)$ can be formally integrated, and with the initial condition $N(0)=\sqrt{Z_2}$ this gives a unique results,

$$N(\lambda)=\Lambda\, e^{-ie^2\int_0^{\lambda}d\lambda\int\frac{d^4k}{(2\pi)^4}\Gamma^{\nu}S'_FD'_{\mu\nu}f^{\mu}\alpha(\lambda)}\cdot\sqrt{Z_2} \tag{49}$$

where: Λ is the ordering operator with respect to λ, and Z_2 is a renormalization constant in the Feynman gauge. The expression for $N(\lambda)$ at $\lambda=1$ is a generalization of the relation (34) for relativistic gauges.

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