DYNAMICAL EQUATIONS FOR SPINLESS, NEUTRAL BOSON SYSTEM, PART II — OFF THE MASS-SHELL FORMULATION

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Functional description of an infinite system of spinless, neutral particles is given in terms of quantities extrapolated off the mass-shell. Quantum Markovian property is postulated and corresponding differential dynamical equations are derived. Finally, the connection with the Rzewuski functional formulation of the quantum field theory is established.

1. Introduction

The present paper is a continuation of our discussion of the quantum causality condition in the context of a theory of spinless, neutral bosons.

In the previous paper on this subject [1], we have formulated a theory in terms of physical quantities namely — the scattering amplitudes on the mass-shell. The dynamical equations describing the time evolution of amplitudes were written there both in differential and integral forms.

In this part we are going to present the simplest possible extrapolation, of the mass--shell of previously obtained results. At the end we discuss the Rzewuski functional formulation of quantum field theory [2], and, in particular, we derive the differential causality condition widely used in this theory.

Naturally, we use exactly the same notation as in the paper [1] to which we will often refer as to I.

2. Off mass shell extrapolation

Let us introduce an auxiliary analytic functional $S'[\sigma, \sigma'; \alpha, \beta]$ by means of the following equation

$$S[\sigma, \sigma'; \alpha, \beta] = e^{\alpha\beta} S'[\sigma, \sigma'; \alpha, \beta] \tag{1.2}$$

where

$$S'[\sigma, \sigma'; \alpha, \beta] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m!\, n!)^{-1/2} \int d\boldsymbol{p}_m \int d\boldsymbol{q}_n \mathscr{S}'(\sigma; \boldsymbol{p}_m|\sigma'; \boldsymbol{q}_n)\, e[\alpha; \boldsymbol{p}_m]\, e[\beta; \boldsymbol{q}_n]. \tag{2.2}$$

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Let $\{\mathscr{S}'(\sigma; p_m|\sigma'; q_n)\}$ be any family of functions depending on the four-vectors p_j and q_k, which agrees with $\{\mathscr{S}'(\sigma; \boldsymbol{p}_m|\sigma'; \boldsymbol{q}_n)\}$ on the mass-shell:

$$\mathscr{S}'(\sigma; \boldsymbol{p}_m|\sigma', \boldsymbol{q}_n)$$
$$= \int dp_m^0 \int dq_n^0 \mathscr{S}'(\sigma; p_m|\sigma'; q_n) \prod_{j=1}^{m} \delta[p_j^0 - \omega(\boldsymbol{p}_j)] \prod_{k=1}^{n} \delta[q_k^0 - \omega(\boldsymbol{q}_k)]. \tag{3.2}$$

We assume, obviously, that the extrapolated amplitudes $\mathscr{S}'(\sigma; p_m|\sigma'; q_n)$ are symmetric in the p_j and q_k variables.

Now we introduce, a set of functions symmetric in x_j and y_k, $\{\mathscr{S}(\sigma; x_m|\sigma'; y_n)\}$ as follows

$$\mathscr{S}(\sigma; x_m|\sigma'; y_n)$$
$$= (-i)^{m+n} (m!\, n!)^{1/2} (2\pi)^{-5/2(m+n)} \int dp_m \int dq_n \prod_{j=1}^{m} [(2\omega(\boldsymbol{p}_j)]^{1/2} \prod_{k=1}^{n} [2\omega(\boldsymbol{q}_k)]^{1/2} \times$$
$$\times \mathscr{S}'(\sigma; p_m|\sigma'; q_n) \exp \{\sum_{j=1}^{m} x_j p_j - i \sum_{k=1}^{n} y_k q_k\}. \tag{4.2}$$

One may easily express the amplitudes $\mathscr{S}'(\sigma; \boldsymbol{p}_m|\sigma'; \boldsymbol{q}_n)$ by these functions as follows

$$\mathscr{S}'(\sigma; \boldsymbol{p}_m|\sigma'; \boldsymbol{q}_n)$$
$$= i^{m+n}(m!\, n!)^{-1/2} \int dx_m \int dy_n \mathscr{S}(\sigma; x_m|\sigma'; y_n) \prod_{j=1}^{m} f(\boldsymbol{p}_j; x_j) \prod_{k=1}^{n} f^*(\boldsymbol{q}_k; y_k) \tag{5.2}$$

where

$$f(\boldsymbol{p}; x) = (2\pi)^{-3/2} \int dp^0\, [2\omega(\boldsymbol{p})]^{-1/2} \cdot \delta[p^0 - \omega(\boldsymbol{p})] \exp(-ipx) \tag{6.2}$$

satisfy the Klein-Gordon equation with a mass m.

It is not difficult to see, [2], that we may impose on $\mathscr{S}(\sigma; x_m|\sigma'; y_n)$ a symmetry condition with respect of exchanging the x_j and y_k variables without affecting the $\mathscr{S}'(\sigma; \boldsymbol{p}_m|\sigma', \boldsymbol{q}_n)$ amplitudes. Namely, if we assume a crossing property,

$$\mathscr{S}(\sigma; \ldots, x_j, \ldots |\sigma'; \ldots, y_k, \ldots) = \mathscr{S}(\sigma; \ldots, y_k, \ldots|\sigma'; \ldots, x_j, \ldots) \tag{7.2}$$

then

$$\mathscr{S}'(\sigma; \ldots, p_j, \ldots|\sigma'; \ldots, q_k, \ldots) = \mathscr{S}'(\sigma; \ldots, -q_k, \ldots|\sigma'; \ldots, -p_j, \ldots). \tag{8.2}$$

For this reason we shall assume that

$$\mathscr{S}(\sigma; x_m|\sigma'; y_n) = \mathscr{S}(\sigma, \sigma'; x_m, y_n) \tag{9.2}$$

$$\mathscr{S}(\sigma, \sigma'; x_1, \ldots, x_n) = \mathscr{S}(\sigma, \sigma'; x_{\pi(1)}, \ldots, x_{\pi(n)}) \tag{10.2}$$

for any permutation $\pi \in S_n$.

According to (1.2), (2.2) and (5.2) we have now

$$S[\sigma, \sigma'; \alpha, \beta]$$
$$= e^{\alpha\beta} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{i^{m+n}}{m!\, n!} \int dx_m \int dy_n \mathscr{S}(\sigma, \sigma'; x_m, y_n) e[\alpha f; x_m] e[\beta f^*; y_n] \tag{11.2}$$

where

$$(\alpha f)(x) = \int d\boldsymbol{p}\, \alpha(\boldsymbol{p}) f(\boldsymbol{p}; x)$$

$$(\beta f^*)(y) = \int d\boldsymbol{q}\, \beta(\boldsymbol{q}) f^*(\boldsymbol{q}; y). \tag{12.2}$$

After rearranging the order of summation in (11.2) we finally get

$$S[\sigma, \sigma'; \alpha, \beta] = e^{\alpha\beta} \sum_{m=0}^{\infty} \frac{i^m}{m!} \int dx_m \mathscr{S}(\sigma, \sigma'; x_m) e[\alpha f + \beta f^*; x_m]. \tag{13.2}$$

Here, $\alpha f + \beta f^*$ according to (6.2), is a general solution of the Klein-Gordon equation

$$(\alpha f + \beta f^*)(x)$$

$$= (2\pi)^{-3/2} \int dp\, \theta(p)\, \delta(p^2 - m^2)\, [2\omega(\boldsymbol{p})]^{1/2}\, [\alpha(\boldsymbol{p})\, e^{-ipx} + \beta(\boldsymbol{p})\, e^{ipx}] \tag{14.2}$$

$$\equiv q_0[\alpha, \beta; x].$$

Therefore we may write

$$S[\sigma, \sigma'; \alpha, \beta] = e^{\alpha\beta} \mathscr{S}[\sigma, \sigma'; q_0[\alpha, \beta]] \tag{15.2}$$

where $\mathscr{S}[\sigma, \sigma'; q]$ is a new generating functional depending of one variable $q(x)$ only,

$$\mathscr{S}[\sigma, \sigma'; q] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_n \mathscr{S}(\sigma, \sigma'; x_n) e[q; x_n]. \tag{16.2}$$

Our next task will consist of expressing the basic postulates in terms of this functional. More exactly, we shall impose upon the generating functional $\mathscr{S}[\sigma, \sigma'; q]$ conditions which will imply the postulates (i—vii, I), satisfied by $S[\sigma, \sigma'; \alpha, \beta]$.

3. *The postulates on* $\mathscr{S}[\sigma, \sigma'; q]$

It is not difficult to verify using (15.2) that the following set of conditions will be appriopriate here:

(i) $\mathscr{S}[\sigma, \sigma'; q]$ — analytic functional

(ii) $$(\mathscr{S}[\sigma, \sigma'; q])^* = \mathscr{S}[\sigma', \sigma; q^*]$$

(iii) $$\lim_{\sigma \to \sigma'} \mathscr{S}[\sigma, \sigma'; q] = 1$$

(iv) $$\mathscr{S}[\sigma, \sigma'; q] \exp\left\{ i \frac{\overleftarrow{\delta}}{\delta q} \Delta^+ \frac{\overrightarrow{\delta}}{\delta q} \right\} \mathscr{S}^*[\sigma, \sigma'; q] = 1$$

for any q,

(v) $$\mathscr{S}[\sigma', \sigma; q] \exp\left\{ i \frac{\overleftarrow{\delta}}{\delta q} \Delta^+ \frac{\overrightarrow{\delta}}{\delta q} \right\} \mathscr{S}[\sigma, \sigma''; q] = \mathscr{S}[\sigma', \sigma''; q]$$

for each q,

$$(vi) \qquad \int dy_n \prod_{k=1}^{n} f^*(p_k; y_k)\, e\left[\frac{\delta}{\delta q}; y_n\right] \mathscr{S}[\sigma, \sigma'; q]|_{q=0} = 0, \qquad n > 0$$

$$\mathscr{S}[\sigma, \sigma'; 0] = 1$$

(vacuum stability), and

$$e\left[f^* \frac{\delta}{\delta q}; q_n\right]\left(f \frac{\delta}{\delta q}\right)(p)\, \mathscr{S}[\sigma, \sigma'; q]|_{q=0} = 0, \qquad n \geqslant 1$$

(one-particle stability condition).

$$\mathscr{S}[\sigma, \sigma'; q] = \mathscr{S}[\sigma + a, \sigma' + a; q_{(a,L)}]$$

$$(vii) \qquad q_{(a,L)}(x) = q[L^{-1}(x-a)].$$

Here the expression $F[q] \exp\left\{i \frac{\overleftarrow{\delta}}{\delta q} \Delta^+ \frac{\overrightarrow{\delta}}{\delta q}\right\} G[q]$ means

$$\exp\left\{i \frac{\delta}{\delta q_1} \Delta^+ \frac{\delta}{\delta q_2}\right\} F[q_1] G[q_2]|_{q_1=q_2=q} \tag{1.3}$$

where

$$\frac{\delta}{\delta q_1} \Delta^+ \frac{\delta}{\delta q_2} = \int dx \int dy \frac{\delta}{\delta q_1(x)} \Delta^+(x-y) \frac{\delta}{\delta q_2(y)} \tag{2.3}$$

$$i\Delta^+(x) = (2\pi)^{-3} \int dp\, \theta(p)\, \delta(p^2 - m^2) \exp(ipx). \tag{3.3}$$

The above set of conditions is the simplest one but is by no means unique. It is possible, for instance, in the unitarity condition (*iv*) to replace the right-hand side by any functional having value one on the mass-shell, [2].

In order to illustrate the method of obtaining these conditions we shall derive the causality condition (*v*) which, in this form, is a new one in quantum field theory.

We start from the causality condition on $S[\sigma, \sigma']$

$$S\left[\sigma', \sigma; \alpha, \frac{\delta}{\delta \varrho}\right] S[\sigma, \sigma''; \varrho, \beta]|_{\varrho=0} = S[\sigma', \sigma''; \alpha, \beta]. \tag{4.3}$$

Using [I, (7.3)] and the formula (15.2) we may write

$$\exp\left\{\alpha \frac{\delta}{\delta \varrho} + q_0\left[\alpha, \frac{\delta}{\delta \varrho}\right] \frac{\delta}{\delta q_1}\right\} \mathscr{S}[\sigma', \sigma; q_1] \exp\left\{\varrho\beta + q_0[\varrho, \beta] \frac{\delta}{\delta q_2}\right\} \mathscr{S}[\sigma, \sigma''; q_2]|_{q_1=q_2=\varrho=0}$$

$$= \exp\left\{\alpha\beta + q_0[\alpha, \beta] \frac{\delta}{\delta q}\right\} \mathscr{S}[\sigma', \sigma''; q]|_{q=0}. \tag{5.3}$$

Performing the translation operation $\exp\left\{\alpha \frac{\delta}{\delta\varrho}\right\}$ we obtain

$$\exp\left\{q_0\left[\alpha_1 \frac{\delta}{\delta\varrho}\right]\frac{\delta}{\delta q_1}\right\}\exp\left\{(\alpha+\varrho)\,\beta+q_0[\alpha+\varrho,\beta]\,\frac{\delta}{\delta q_2}\right\}\Big|_{\varrho=0}\mathscr{S}[\sigma',\sigma;q_1]\mathscr{S}[\sigma,\sigma'';q_2]|_{q_1=q_2=0}$$

$$=\exp\left\{\alpha\beta+q_0[\alpha,\beta]\,\frac{\delta}{\delta q}\right\}\mathscr{S}[\sigma',\sigma'';q]|_{q=0}. \tag{6.3}$$

Now we remark that the ϱ-variable enters this formula only in the $\alpha+\varrho$ combination. Hence, we may replace the differentiation operation $q_0\left[\alpha,\frac{\delta}{\delta\varrho}\right]$ by $q_0\left[\alpha,\frac{\vec{\delta}}{\delta\alpha}\right]$ where the arrow indicates that only α's standing outside on the right should be differentiated. We have therefore

$$\exp\left\{q_0\left[\alpha',\frac{\delta}{\delta\alpha}\right]\frac{\delta}{\delta q_1}\right\}\exp\left\{\alpha\beta+q_0[\alpha,\beta]\frac{\delta}{\delta q_2}\right\}\mathscr{S}[\sigma',\sigma;q_1]\mathscr{S}[\sigma,\sigma'';q_2]|_{\substack{q_1=q_2=0\\ \alpha'=\alpha}}$$

$$=\exp\left\{\alpha\beta+q_0[\alpha,\beta]\frac{\delta}{\delta q}\right\}\mathscr{S}[\sigma',\sigma'';q]|_{q=0}. \tag{7.3}$$

Using the formula

$$F\left[\frac{\delta}{\delta\alpha}\right]\exp(\alpha\beta)=\exp(\alpha\beta)\,F\left[\beta+\frac{\delta}{\delta\alpha}\right] \tag{8.3}$$

which is valid for any analytic functional $F[\alpha]$, we extract the factor $\exp(\alpha\beta)$ and cancel it on both sides

$$\exp\left\{q_0\left[\alpha,\beta+\frac{\vec{\delta}}{\delta\alpha}\right]\right\}\exp\left\{q_0[\alpha,\beta]\frac{\delta}{\delta q_2}\right\}\mathscr{S}[\sigma',\sigma;q_1]\mathscr{S}[\sigma,\sigma'';q_2]|_{q_1=q_2=0}$$

$$=\exp\left\{q_0[\alpha,\beta]\frac{\delta}{\delta q}\right\}\mathscr{S}[\sigma',\sigma'';q]|_{q=0}. \tag{9.3}$$

Using once more the formula [I. (7.3)] we may write

$$\exp\left\{q_0\left[\alpha,\beta+\frac{\vec{\delta}}{\delta\alpha}\right]\frac{\delta}{\delta q_1}\right\}=\exp\left\{q_0[\alpha,\beta]\frac{\delta}{\delta q_1}\right\}\exp\left\{\frac{\overleftarrow{\delta}}{\delta\beta}\frac{\vec{\delta}}{\delta\alpha}\right\} \tag{10.3}$$

where

$$\frac{\overleftarrow{\delta}}{\delta\beta(\mathbf{p})}=\int dx\,\frac{\delta q_0[\alpha,\beta;x]}{\delta\beta(\mathbf{p})}\,\frac{\overleftarrow{\delta}}{\delta q_0(x)}=\int dx f^*(\mathbf{p};x)\,\frac{\overleftarrow{\delta}}{\delta q_0(x)}$$

$$\frac{\vec{\delta}}{\delta\alpha(\mathbf{p})}\int dy\,\frac{\delta q_0[\alpha,\beta;y]}{\delta\alpha(\mathbf{p})}\cdot\frac{\vec{\delta}}{\delta q_0(y)}=\int dy\,f(\mathbf{p};y)\,\frac{\vec{\delta}}{\delta q_0(y)}\,. \tag{11.3}$$

Due to the completeness relation we have

$$i\Delta^+(x-y)=\int d\mathbf{p} f^*(\mathbf{p};x)f(\mathbf{p};y) \tag{12.3}$$

$$\frac{\overleftarrow{\delta}}{\delta\beta}\frac{\vec{\delta}}{\delta\alpha}=i\frac{\overleftarrow{\delta}}{\delta q_0}\Delta^+\frac{\vec{\delta}}{\delta q_0}\,. \tag{13.3}$$

Utilizing the above facts we may rewrite the formula (9.3) as follows

$$\exp\left\{q_0[\alpha, \beta]\frac{\delta}{\delta q_1}\right\}\exp\left\{i\frac{\overleftarrow{\delta}}{\delta q_0}\Delta^+\frac{\overrightarrow{\delta}}{\delta q_0}\right\}\exp\left\{q_0[\alpha, \beta]\frac{\delta}{\delta q_2}\right\}\mathscr{S}[\sigma', \sigma; q_1]\mathscr{S}[\sigma, \sigma''; q_2]|_{q_1=q_2=0}$$

$$= \exp\left\{q_0[\alpha, \beta]\frac{\delta}{\delta q}\right\}\mathscr{S}[\sigma', \sigma''; q]|_{q=0}. \tag{14.3}$$

Finally, using again the formula [I. (7.3)] we obtain

$$\mathscr{S}[\sigma', \sigma; q_1+q_0[\alpha, \beta]]\exp\left\{i\frac{\delta}{\delta q_1}\Delta^+\frac{\delta}{\delta q_2}\right\}\mathscr{S}[\sigma, \sigma''; q_2+q_0[\alpha, \beta]]|_{q_1=q_2=0} =$$

$$=\mathscr{S}[\sigma', \sigma''; q+q_0[\alpha, \beta]]|_{q=0}. \tag{15.3}$$

This condition will certainly be satisfied if the following relation holds identically in q

$$\mathscr{S}[\sigma', \sigma; q]\exp\left\{i\frac{\overleftarrow{\delta}}{\delta q}\Delta^+\frac{\overrightarrow{\delta}}{\delta q}\right\}\mathscr{S}[\sigma, \sigma''; q] = \mathscr{S}[\sigma', \sigma''; q]. \tag{16.3}$$

For $q = q_0[\alpha, \beta]$ we obtain the mass-shell causality condition.

For the sake of convenience we introduce the abbreviation

$$\exp\left\{i\frac{\overleftarrow{\delta}}{\delta q}\Delta^+\frac{\overrightarrow{\delta}}{\delta q}\right\} \equiv (*) \tag{17.3}$$

where brackets serve to distinguish this multiplication prescription from the multiplication law of functional matrices on the mass-shell, [*cf.*, I. (3.4)].

4. *Dynamical equations and Hamiltonian off the mass-shell*

Let us start the Hamiltonian on the mass-shell, [*cf.*, I. (18.5) and I. (8.5)],

$$H[\sigma'; y; \alpha, \beta] = i\hbar S^+[\sigma, \sigma'] * \frac{\delta S[\sigma, \sigma']}{\delta\sigma'(y)}[\alpha, \beta] = i\hbar S\left[\sigma', \sigma; \alpha, \frac{\delta}{\delta\varrho}\right]\frac{\delta S[\sigma, \sigma'; \varrho, \beta]}{\delta\sigma'(y)}\bigg|_{\varrho=0}. \tag{1.4}$$

Using the definition of $\mathscr{S}[\sigma, \sigma'; q]$, (15.2), we obtain

$$H[\sigma'; y; \alpha, \beta]$$

$$= e^{\alpha\beta}i\hbar\mathscr{S}[\sigma', \sigma; q+q_0[\alpha, \beta]]\,(*)\,\frac{\delta\mathscr{S}[\sigma, \sigma'; q+q_0[\alpha, \beta]]}{\delta\sigma'(y)}\bigg|_{q=0}$$

$$= e^{\alpha\beta}\mathscr{H}[\sigma'; y; q+q_0[\alpha, \beta]]|_{q=0} \tag{2.4}$$

where $\mathscr{H}[\sigma'; y; q]$ is the off mass-shell Hamiltonian

$$\mathscr{H}[\sigma'; y; q] = i\hbar\mathscr{S}[\sigma', \sigma; q]\,(*)\,\frac{\delta\mathscr{S}[\sigma, \sigma'; q]}{\delta\sigma'(y)}. \tag{3.4}$$

In the same way as in the Chapter I.5 one may show that $\mathscr{H}$ does not depend on σ and the formula holds

$$\mathscr{H}[\sigma'; y; q] = i\hbar\mathscr{S}^{-1}[\sigma'; q]\,(*)\,\frac{\delta\mathscr{S}[\sigma'; q]}{\delta\sigma'(y)} \tag{4.4}$$

where

$$\mathscr{S}[\sigma; q] = \lim_{\sigma' \to -\infty} \mathscr{S}[\sigma', \sigma; q]. \tag{5.4}$$

The most natural extrapolations of the dynamical equations I. (19.5) are the following ones

$$i\hbar \frac{\delta \mathscr{S}[\sigma, \sigma'; q]}{\delta \sigma'(y)} = \mathscr{S}[\sigma, \sigma'; q]\,(*)\mathscr{H}[\sigma'; y; q] \tag{6.4}$$

$$-i\hbar \frac{\delta \mathscr{S}[\sigma, \sigma'; q]}{\delta \sigma(x)} = \mathscr{H}[\sigma; x; q]\,(*)\mathscr{S}[\sigma, \sigma'; q]. \tag{7.4}$$

They can be obtained from I. (19.5) by the same procedure of extrapolation as *e. g.*, Hamiltonian $\mathscr{H}$ was derived.

5. *Rzewuski's generating functional*

In this section we will show that the theory proposed by Rzewuski [2], [3] is contained in the present one. The main differences between these formulations follow from the facts that the Rzewuski theory does not include the causality condition on the mass shell, I. (v), nor the motion reversal invariance condition. Instead, the Bogoliubov type [4], differential causality condition is imposed on the off mass-shell extrapolated generating functional $\Omega[q]$ which in addition, is a non- σ depending quantity.

In order to obtain this theory we assume that

$$\mathscr{S}[\sigma, \sigma'; q] = \Omega[q_{\sigma\sigma'}] \tag{1.5}$$

where $\Omega[q]$ is some analytic functional

$$\Omega[q] = \sum_{m=0}^{\infty} \frac{i^m}{m!} \int dx_m \omega(x_m) e[q; x_m] \tag{2.5}$$

and $q_{\sigma\sigma'}$ is the restriction of $q(x)$ to the $[\sigma, \sigma']$ interval *i.e.*,

$$q_{\sigma\sigma'}(x) = \begin{cases} q(x) \text{ if } \sigma \leqslant x \leqslant \sigma' \\ 0 \text{ outside.} \end{cases}$$

Clearly $\Omega[q]$, where q is not restricted, is a generating functional for the matrix elements of the conventional S matrix

$$\Omega[q] = \lim_{\substack{\sigma' \to \infty \\ \sigma \to \infty}} \mathscr{S}[\sigma, \sigma'; q] \tag{4.5}$$

$$S[\alpha, \beta] = e^{\alpha\beta} \Omega[q_0^{[\alpha,\beta]}]. \tag{5.5}$$

The following conditions on $\Omega[q]$ are implied by the postulates (i–vii) for $\mathscr{S}[\sigma, \sigma'; q]$

(i) $\Omega[q]$ — analytic functional

(ii) $(\Omega[q])^* = \Omega^*[q^*]$ always fulfilled

(*iii*)
$$\lim_{\sigma\to\sigma'} \Omega\,[q_{\sigma\sigma'}] = \Omega\,[0] = 1$$

(*iv*)
$$\Omega(q)\,(*)\,(\Omega[q])^* = 1$$

(*v*)
$$\Omega[q_{\sigma'\sigma}]\,(*)\,\Omega[q_{\sigma\sigma''}] = \Omega[q_{\sigma'\sigma''}] \qquad \sigma' < \sigma < \sigma''$$

(*vi*)
$$e\left[f^*\frac{\delta}{\delta q};\,\mathbf{q}_n\right]\Omega[q]|_{q=0} = 0, \qquad n > 0$$

$$e\left[f^*\frac{\delta}{\delta q};\,\mathbf{q}_n\right]f\frac{\delta}{\delta q}(\mathbf{p})\Omega[q]|_{q=0} = 0, \qquad n \geqslant 0$$

(*vii*)
$$\Omega(q) = \Omega[q_{(a,L)}]$$

$$q_{(a,L)}(x) = q[L^{-1}(x-a)],\ L\in L_+^\uparrow(R).$$

In order to derive the differential causality condition from (*v*) we proceed in the following way (*cf. e. g.*, [4], § 17);

We decompose a real $q(x)$ into two parts

$$q(x) = q_{(-\infty,\sigma)}(x) + q_{(\sigma,\infty)}(x) \tag{6.5}$$

and substitute into the equation (*v*)

$$\Omega[q] = \Omega[q_{(-\infty,\sigma)}]\,(*)\,\Omega[q_{(\sigma,\infty)}]. \tag{7.5}$$

Here, surface σ is a plane orthogonal to the time axis.

Consider now the two functions

$$q'(x) = \underset{(-\infty,\sigma)}{q'(x)} + \underset{(\sigma,\infty)}{q(x)} \tag{8.5}$$

$$q''(x) = \underset{(-\infty,\sigma)}{q''(x)} + \underset{(\sigma,\infty)}{q(x)}$$

and calculate the expression

$$Q(q', q''; \sigma) = \Omega[q']\,(*)\,\Omega^*[q'']. \tag{9.5}$$

Because of the general formula

$$(\Omega_1[q]\,(*)\,\Omega_2[q])^* = \Omega_2^*[q]\,(*)\,\Omega_1^*[q] \tag{10.5}$$

we obtain easily

$$Q(q', q''; \sigma) = \Omega[\underset{(-\infty,\sigma)}{q'}]\,(*)\,\Omega^*[\underset{(-\infty,\sigma)}{q''}]. \tag{11.5}$$

This formula shows that $Q(q', q''; \sigma)$ does not depend on q in the region of space-time later than σ.

Therefore, if one chooses

$$q'(x) = q(x)$$

$$q''(x) = q(x) + \delta q(x) \tag{12.5}$$

where $\delta q(x)$ is localized in the region earlier than σ one obtains

$$Q(q, q+\delta q; \sigma) = \Omega[q_{(-\infty,\sigma)}](*)\, \Omega^*[q_{(-\infty,\sigma)} + \delta q]$$

$$= \Omega[q_{(-\infty,\sigma)}]\, (*)\, \{\Omega^*[q_{(-\infty,\sigma)}] + \delta\Omega^*[q_{(-\infty,\sigma)}] + 0(\delta q)\}$$

$$= 1 + \Omega[q_{(-\infty,\sigma)}]\, (*)\, \delta\Omega^*[q_{(-\infty,\sigma)}] + 0(\delta q) \tag{13.5}$$

where

$$\delta\Omega[q_{(-\infty,\sigma)}] = \int\limits_{x^0<\sigma} dx\, \frac{\delta\Omega[q]}{\delta q(x)}\, \delta q(x). \tag{14.5}$$

The fact of independence of $Q(q, q+\delta q; \sigma)$ on the behaviour of $q(x)$ at $x_0 > \sigma$ gives us

$$\frac{\delta Q(q, q+\delta q; \sigma)}{\delta q(y)} = 0, \quad y^0 > \sigma \tag{15.5}$$

i. e.,

$$\int\limits_{x^0<\sigma} dx \frac{\delta}{\delta q(y)} \left\{\Omega[q]\, (*) \left(\frac{\delta\Omega[q]}{\delta q(x)}\right)^*\right\} \delta q(x) = 0, \quad y^0 > \sigma \tag{16.5}$$

for $y_0 > \sigma$.

Since $\delta q(x)$ is an arbitrary variation in the region $x_0 < \sigma$ we conclude

$$C[q; y, x] = \frac{\delta}{\delta q(y)} \left\{\Omega[q]\, (*) \left(\frac{\delta\Omega[q]}{\delta q(x)}\right)^*\right\} = 0 \quad \text{for } x^0 < \sigma < y^0. \tag{17.5}$$

In the limit $x_0 \to \sigma$ we obtain a condition $x^0 < y^0$ for the above equation to be valid. Due to Lorentz invariance we have from this equation for any $L \in L_+^\uparrow(R)$

$$C[q; y, x] = C[q_L; Ly, Lx]. \tag{18.5}$$

Since for any pair x, y of vectors such that $x-y$ is space-like there exists a proper Lorentz transformation L such that $(Lx)^0 < (Ly)^0$, we finally conclude

$$\frac{\delta}{\delta q(y)} \left\{\Omega[q]\, (*) \left(\frac{\delta\Omega[q]}{\delta q(x)}\right)^*\right\} = 0 \quad x \lesssim y \tag{19.5}$$

where $x \sim y$ means an $x-y$ space-like vector.

This is the demanded differential causality condition, which was, in fact, postulated by Rzewuski [2].

6. *Concluding remarks*

From the above brief considerations one sees that the quantum causality condition (*v*) is a really powerful one and is mainly responsible for the dynamics of a system. Therefore, it should be possible to derive from it all the really relevant analytic properties of scattering amplitudes which are usually postulated, without good justification, in the phenomenological

S-matrix theories, [5]. Moreover, the present theory may also be regarded as a general rule of constructing phenomenological models which one gets by restricting the summation over the number of intermediate particles in the causality and unitarity conditions to one particle, two particle etc., cases. All such questions deserve special investigations and will be considered elsewhere.

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