ELECTROMECHANICAL RESPONSE IN AN INHOMOGENEOUS PIEZOELECTRIC TRANSDUCER SUBJECTED TO A STATIC CHARGE

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The present note is concerned with the electrical and mechanical response in an inhomogeneous and rigidly backed piezoelectric transducer subjected to a static charge.

1. Introduction

The studies of responses — electrical or mechanical in a piezoelectric transducer are of very recent origin. The important works that should be mentioned in this case are those of Redwood [1], Mason [2], Fillipczynski [3], Holland [4], Sinha [5], [6], Kaliski [7] and others. The papers of Stuetzer [8], Das [9], Giri [10], Roy [11], Chakrabarty [12] on transducers should also be referred to. In all these papers the transducer considered is homogeneous in regard to its material parameters. It is well known that crystals, particularly the piezoelectric ones contain impurities, vide, Sinha [13] which render the crystal inhomogeneous. It therefore seems worthwhile to investigate the disturbances in an inhomogeneous transducer owing to assigned inputs.The types of inhomogeneities in isotropic and anisotropic materials, it may be recalled, have been amply discussed by Olszak [14]. The present note is concerned with the electromechanical responses in an inhomogeneous piezoelectric plate transducer when a static charge is suddenly developed across its ends, the transducer being open circuited as shown by Stuetzer [8], the kind of inhomogeneity being one of those suggested by Olszak [14]. As is usual with such problems, Laplace transform is used to achieve the solution of the problem.

2. Problem, fundamental equation and boundary conditions

We consider here a piezoelectric transducer (a mechanism to convert electrical energy to mechanical energy and vice versa) in the form of a plate executing disturbances in the direction of its thickness which, *vide* Redwood [1] is the mode of vibration for generation of ultrasonic waves. We take the X-axis in the direction of thickness and let $x = 0$, $x = X$

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be the extremities of the transducer. We further assume that the Y and Z dimensions of the transducer are much greater than the thickness dimension, so that as in Redwood [1] we can suppose the disturbances to propagate undisturbed in the direction of X-axis. In essence this reduces the problem to a one dimensional one. The material paramaters of the transducer are inhomogeneous. A possible kind of inhomogeneity is the exponential dependence of the parameters on the dimensions, *vide* Olszak [14]. Across the plate a static charge Q_0 is suddenly developed. The electromechanical coupling will give rise to a voltage. Our aim is to evaluate the expression for this voltage and the mechanical displacement at any point within the transducer.

The constitutive equations for the transducer, are, *vide* Kaymme [15],

$$T_1 = c_{11} \frac{\partial \xi}{\partial x} + e_{11} E_1 \tag{1}$$

$$D_1 = \varepsilon E_1 + 4\pi e_{11} \frac{\partial \xi}{\partial x} \tag{2}$$

where T_1 is stress, c_{11} is average stiffness coefficient, ξ is the mechanical displacement at any point x, e_{11} is piezoelectric constant, E_1 is electric field, D_1 is the electric displacement along the axial direction and ε is the dielectric permittivity, c_{11}, e_{11} and ε are choosen in the following way:

$$c_{11} = c_{11}^0 e^{kx} \tag{3}$$

$$e_{11} = e_{11}^0 e^{kx} \tag{4}$$

$$\varepsilon = \varepsilon^0 e^{kx} \tag{5}$$

Eliminating E_1 from (1) and (2) with the aid of (3), (4) and (5), we have

$$T_1 = a e^{kx} \cdot \frac{\partial \xi}{\partial x} + \frac{e_{11}^0}{\varepsilon^0} D_1 \tag{6}$$

where

$$a = c_{11}^0 - \frac{4\pi e_{11}^{02}}{\varepsilon^0}.$$

Since plane distrurbance propagation is assumed, differentials with respect to y and z are zero, and in particular

$$\frac{\partial D_y}{\partial y} = \frac{\partial D_z}{\partial z} = 0. \tag{6A}$$

Also from Gauss's Law, since there is no free charge inside the transducer,

$$\nabla \cdot \overline{D} = 0 \tag{6B}$$

Hence from (6A) and (6B) we have

$$\frac{\partial D_1}{\partial x} = 0. \tag{6C}$$

Differentianting (6) and making use of (6C) we get

$$\frac{\partial T_1}{\partial x} = e^{kx}\left[a\frac{\partial^2\xi}{\partial x^2} + ka\frac{\partial\xi}{\partial x}\right] \tag{7}$$

(7) along with Newton's equation

$$\frac{\partial T_1}{\partial x} = \varrho_0 e^{kx}\frac{\partial^2\xi}{\partial t^2}$$

(ϱ is assumed to be $= \varrho_0 e^{kx}$) yields

$$a\frac{\partial^2\xi}{\partial x^2} + ka\frac{\partial\xi}{\partial x} - \varrho_0\frac{\partial^2\xi}{\partial t^2} = 0. \tag{8}$$

Equation (8) is the fundamental equation of our problem. The boundary conditions, expressed mathematically, are given by:

1) since the transducer is rigidly backed at $x = X$,

$$\xi(X) = 0$$

2) the stresses and displacements are continuous at $x = 0$.

3. *Solution of the problem*

Taking Laplace transform of (8) of parameter $s(s > 0)$ we have

$$a\frac{\partial^2\bar{\xi}}{\partial x^2} + ka\frac{\partial\bar{\xi}}{\partial x} - \varrho_0 s^2\bar{\xi} = 0 \tag{9}$$

where

$$\xi(0) = 0 \quad \text{and} \quad \xi'(0) = 0.$$

Solving (9) we get

$$\bar{\xi} = Ae^{m_2 x} + Be^{m_1 x} \tag{10}$$

where

$$m_1, m_2 = \frac{-ka \pm \sqrt{k^2a^2 + 4a\varrho_0 s^2}}{2a} \tag{11}$$

To evaluate the constants A and B we attach two mechanical systems 1 and 2 as in Redwood [1] to the extremities $x = 0$ and $x = X$. We denote the corresponding entities (*i.e.* A_1, B_1 and A_2, B_2) which are represented by the symbols 1 and 2 respectively. In that case, since the transducer is rigidly backed,

$$A_2 = B_2 = 0 \quad \text{and} \quad A_1 = 0 \tag{12}$$

The continuity of displacement at $x = 0$, gives

$$\bar{\xi}(0) = \bar{\xi}_1(0)$$

$$\text{i.e.} \quad A + B = B_1 \tag{13}$$

where

$$\bar{\xi}_1 = B_1 e^{m_1 x} \tag{14}$$

Also $\bar{\xi}(X) = 0$ for rigidly backed. This yields

$$Ae^{m_2 X} + Be^{m_1 X} = 0 \tag{15}$$

Taking Laplace transform of (6) and using (10) we have

$$\bar{T}_1(0) = a[m_2 A + m_1 B] + \frac{e_{11}^0}{\varepsilon^0} \cdot \frac{Q}{YZ} \cdot \frac{1}{s}. \tag{16}$$

If T stands for the stress for the medium towards the left of the transducer (*i. e.* at $x = 0$) then from (14) we get

$$\bar{T}(0) = \mathrm{c} m_1 B_1 \tag{17}$$

c being a similar constant as c_{11}.

For continuity of stress at $x = 0$ we have from (16) and (17),

$$a[m_2 A + m_1 B) + \frac{e_{11}^0}{\varepsilon^0} \frac{Q_0}{YZ} \cdot \frac{1}{s} = cm_1 B_1. \tag{18}$$

Solving for A and B, from (13), (15) and (18) we get

$$A = \frac{-l_3 e^{m_1 X}}{l_1 e^{m_1 X} - l_2 e^{m_2 X}} \quad \text{and} \quad B = \frac{l_3 e^{m_2 X}}{l_1 e^{m_1 X} - l_2 e^{m_2 X}}$$

where

$$l_1 = am_2 - \mathrm{c}m_1$$

$$l_2 = m_1 (a - c) \tag{19}$$

$$l_3 = \frac{e_{11}^0 Q_0}{\varepsilon^0 YZ} \cdot \frac{1}{s}$$

Substituting the value of A and B in (10) we have

$$\bar{\xi} = \frac{l_3 [e^{m_2 X} \cdot e^{m_1 X} - e^{m_1 X} \cdot e^{m_2 X}]}{l_1 e^{m_1 X} - l_2 e^{m_2 X}}. \tag{20}$$

Relation (20) gives $\bar{\xi} = 0$ when $x = X$. This is in accordance with our assumption.

To evaluate ξ we take the inverse transform of (2). The relation being complicated we adopt the method of approximation as in Redwood [1]. For simplicity we will calculate ξ for $x = X/2$. To a first degree of approximation, as in Redwood [1],

$$\xi_{\frac{X}{2}} \approx \left(\alpha \cdot \frac{1}{s^2} + \beta \cdot \frac{1}{s^3}\right) e^{-bs} \cdot e^{-\frac{d}{s}} \tag{21}$$

where

$$\alpha = \frac{e_{11}^0 Q_0 \sqrt{a}}{\varepsilon^0 YZ \sqrt{\varrho_0} (a + c)} e^{-\frac{K}{2}}, \quad \beta = \frac{e_{11}^0 Q_0}{\varepsilon^0 YZ} \cdot \frac{a(a-c)K}{2(a+c)^2 \varrho_0} \cdot e^{-\frac{K}{2}}$$

$$b = \sqrt{\frac{\varrho_0}{a}} \quad \text{and} \quad d = \frac{K^2}{8} \sqrt{\frac{a}{\varrho_0}}. \tag{22}$$

Taking the inverse transform of (21) we have, when $t > b$

$$\xi_{\frac{X}{2}} = \alpha \cdot \int_0^t J_0\{2\sqrt{d(t-u)}\} \cdot du + \beta \int_0^t \sqrt{\frac{t-u}{d}} \cdot J_1\{2\sqrt{d(t-u)}\} du \tag{23}$$

where J_n is Bessel function of first kind and order n.
and

$$\xi_{\frac{X}{2}} = 0 \text{ when } t < b \tag{24}$$

The inverse transform is carried out by the well-known convolution theorem, *vide* Churchill [16]. Relation (23) and (24) gives the expression for the mechanical displacement in terms of the given parameters.

Electrical response:

The electrical response is found out by using the relation,

$$\bar{V} = -(\bar{V}_x - \bar{V}_0) = \int_0^X \bar{E}, dx \tag{25}$$

Making use of (2), (4), (5) and noting $\bar{D}_1 = \frac{\bar{Q}_1}{YZ} = \frac{Q_0}{YZ} \cdot \frac{1}{s}$ we have from (25)

$$\bar{V} = -\left[\frac{e^{-KX}}{K} + 1\right] \frac{Q_0}{\varepsilon^0 YZ} \cdot \frac{1}{s} + \frac{4\pi e_{11}^0}{\varepsilon_{11}^0} (A+B).$$

Substituting for A and B and taking approximation as in Redwood [1] we have

$$\bar{V} = -\left[\frac{e^{-KX}}{K} + 1\right] \frac{Q_0}{\varepsilon^0 YZ} \cdot \frac{1}{s} - \frac{4\pi e_{11}^{0^2}}{\varepsilon^{0^2}} \cdot \frac{Q_0}{YZ} \cdot \frac{1}{s} \left[\frac{\sqrt{a}}{\sqrt{\varrho_0}(a+c)} \cdot \frac{1}{s} + \frac{a(a-c)K}{2\varrho_0(a+c)^2} \cdot \frac{1}{s^2}\right].$$

Taking the inverse transform,

$$V = -\left[\frac{e^{-KX}}{K} + 1\right] \frac{Q_0}{\varepsilon^0 YZ} - \frac{4\pi e_{11}^{0^2}}{\varepsilon^{0^2}} \cdot \frac{Q_0}{YZ} \sqrt{\frac{a}{\varrho_0}} \cdot \frac{1}{a+c} t - \frac{4\pi e_{11}^{0^2} Q_0 a(a-c)K}{\varepsilon^{0^2} YZ \varrho_0 (a+c)^2} \cdot t^2. \tag{26}$$

This shows that the electrical response is partly constant, partly linear and partly quadratic with time.

4. *Discussion*

In the absence of inhomogeneous terms, we can show that

$$\left.\begin{aligned} \xi_{\frac{X}{2}} &= \frac{bc}{\varrho} t^2 + \psi_1 \cdot \left(t - \frac{X}{2v}\right) + \psi_2 \cdot \left(t - \frac{3X}{2v}\right) && \text{when } t > \frac{3X}{2v} \\ &= \frac{bc}{\varrho} t^2 + \psi_1 \cdot \left(t - \frac{X}{2v}\right) && \text{when } \frac{X}{2v} < t < \frac{3X}{2v} \\ &= \frac{bc}{\varrho} t^2 && \text{when } t < \frac{X}{2v} \end{aligned}\right\} \tag{27}$$

ψ_1 abd ψ_2 are in terms of the given parameters

$$\left.\begin{aligned} V &= \alpha+\beta t+\gamma\left(t-\frac{X}{v}\right) && \text{when} \quad t>\frac{X}{v} \\ &= \alpha+\beta t && \qquad\quad t<\frac{X}{v} \end{aligned}\right\} \tag{28}$$

A comparative study of (23), (24) with (27) and (26) with (28) elucidates the effect of the inhomogeneous nature of the transducer.

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