Proceedings of the 11th Workshop on Quantum Chaos and Localisation Phenomena (CHAOS 23)

Numerical Simulation of Radiative Transfer of Electromagnetic Angular Momentum

B.A. van Tiggelen* and R. Le Fournis

Univ. Grenoble Alpes, CNRS, LPMMC, 38000 Grenoble, France

Doi: 10.12693/APhysPolA.144.495 *e-mail: Bart.Van-Tiggelen@lpmmc.cnrs.fr

We present numerical simulations of the light emitted by a source and scattered by surrounding electric dipoles with Zeeman splitting. We calculate the leakage of electromagnetic angular momentum to infinity.

topics: radiative transfer, magneto-optics

1. Introduction

Optical sources radiate electromagnetic energy at a rate that depends on the local density of radiative states (LDOS) near the source and at the emitted frequency of the source [1, 2]. This statement is classically true and recalls Fermi's golden rule in quantum mechanics [3]. LDOS is affected by the environment and can be either dielectric, structured, gapped in frequency, or disordered. If the environment is magneto-active, induced by the presence of an externally applied magnetic field $\boldsymbol{B}_0$, the source can also radiate angular momentum (AM) with direction $\boldsymbol{B}_0$ into space.

The first study of this phenomenon [4] used the phenomenological concept of radiative transfer and, in particular, the role of the radiative boundary layer of an optically thick medium to argue that the Poynting vector has two components in the far field. The first is the usual energy flux, purely radial, that decays with distance r from the object as $1/r^2$. The second is magneto-transverse and circulates energy around the object (see Fig. 1). This component decays faster, as $1/r^3$, but has finite angular momentum constant with distance that travels away from the object. Our second study [5] demonstrated that this leakage is not restricted to multiple scattering and also exists when a homogeneous magneto-birefringent environment surrounds the source. In this case: (i) the radiation of AM results in a torque on the source and not on the environment, (ii) it depends sensitively on the nature of the source with huge differences between, e.g., an electric dipole source and a magnetic dipole source; and (iii) geometric "Mie" resonances can enhance the effect much like the Purcell effect does in nano-antennas [6, 7].

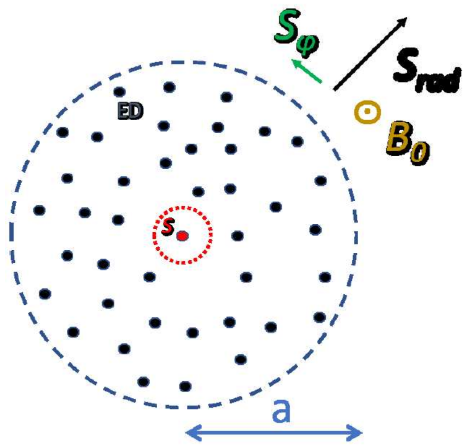

Fig. 1. The geometry considered in this work. A source emits light into a disordered environment containing N electric dipole scatterers. In the presence of a magnetic field $\boldsymbol{B}_0$, a magneto-transverse component $\boldsymbol{S}_\phi$ of the Poynting vector appears outside the medium that carries electromagnetic angular momentum. This angular momentum propagates to infinity, and a torque is exerted on scatterers and source.

Our latest study [8] considered numerically a spherical environment filled with small resonant electric dipole scatterers. When the optical thickness increases at a fixed frequency, the total leakage rate of AM is seen to increase. Upon varying the optical thickness, we investigated separately the role of photonic spin and orbital momentum, the two essential constituents of electromagnetic AM [9], and found both to co-exist. The torque on the source was also seen to increase with dipole density but hardly with optical thickness. In this work, we study the frequency dependence of the dipole scatterers. Especially for large detuning from their resonance, the scattering from one dipole becomes weak, so to keep reasonable optical thicknesses, we require more dipoles, typically thousands, and this takes more CPU time and memory.

2. Leakage of angular momentum

For a monochromatic electric dipole source with electric dipole moment $\boldsymbol{d}$ at frequency $\omega = kc_0$, positioned at $\boldsymbol{r} = 0$, the radiated electric field at position $\boldsymbol{r}$ is given by

$$\boldsymbol{E}(\boldsymbol{r},\omega) = -4\pi k^2\, \boldsymbol{G}\big(\boldsymbol{r},0,\omega{+}\mathrm{i}0\big)\cdot \boldsymbol{d}(\omega) \tag{1}$$

with $G_{kk'}(\boldsymbol{r},\boldsymbol{r}',\omega)$, the vector Green's function, associated with the Helmholtz equation for the electric field. This Green's function contains full information about the environment. The slightly positive imaginary part of the frequency $\omega + \mathrm{i}0$ guarantees outward propagation of the light. The power P (radiated energy per second) radiated by the electric dipole is equal to its dissipation rate $\mathrm{Re}(\boldsymbol{J}^*\cdot\boldsymbol{E})/2$ [10]. Since $\boldsymbol{J} = -\mathrm{i}\omega\boldsymbol{d}$, we find, after averaging over the orientation of the dipole source, that

$$P = -\frac{2\pi}{3}k^3 c_0 \big|\boldsymbol{d}\big|^2\, \mathrm{Im}\Big[\mathrm{Tr}\,\boldsymbol{G}(0,0)\Big]. \tag{2}$$

We recognize $\rho(k) \sim -\mathrm{Im}[\,\mathrm{Tr}\,\boldsymbol{G}(0,0)]$ as LDOS at the source position. The balance equation for the angular momentum can be written as [5]

$$\frac{\mathrm{d}}{\mathrm{d}t}J_{i,\mathrm{mec}} = M_i = \frac{R^3}{8\pi}\epsilon_{ijk}\,\mathrm{Re}\left[\int_{4\pi}\mathrm{d}\hat{\boldsymbol{r}}\ \hat{r}_l\hat{r}_j\big(E_l^*E_k + B_l^*B_k\big)(R\hat{\boldsymbol{r}})\right], \tag{3}$$

with $\boldsymbol{J}_{\mathrm{mec}}$ being the mechanical AM of the matter and with implicit summation over repeated indices. This formula expresses that the torque $\boldsymbol{M}$ exerted on the matter is radiated away as AM to infinity ($r > R$), thereby assuming a source that has been constant during a time longer than R/c_0. In this picture, the radiative AM inside the environment enclosed by the sphere of radius R is constant in time, and AM leaks to infinity somewhere around $r(t) \sim c_0 t > R$. The cycle-averaged torque acts on both the source and its environment, $\boldsymbol{M} = \boldsymbol{M}_S + \boldsymbol{M}_E$. The latter is

$$\boldsymbol{M}_E = \frac{1}{2}\mathrm{Re}\left[\int \mathrm{d}^3r\ (\boldsymbol{P}^*\times\boldsymbol{E} + P_m^*(\boldsymbol{r}\times\nabla)E_m)\right]. \tag{4}$$

This torque vanishes for a rotationally-invariant environment around the source but not when this symmetry is broken by structural heterogeneity, as will be discussed here. The torque on a source with an electric dipole moment $\boldsymbol{d}(\omega)$ is given by [10]

$$\boldsymbol{M}_S = \frac{1}{2}\mathrm{Re}\big[(\boldsymbol{d}^*\times\boldsymbol{E})\big]. \tag{5}$$

With the electric field given by (1), an expression similar to (2) can be obtained, i.e.,

$$M_{S,i} = -\frac{2\pi}{3}k^2\big|\boldsymbol{d}\big|^2\ \mathrm{Re}\Big[\epsilon_{ijk}G_{kj}(0,0)\Big]. \tag{6}$$

This torque vanishes for an (on average) spherical environment with isotropic optical response [11].

Alternatively, the leak of AM given by the right-hand side of (3) can be split up into parts associated with photonic spin and orbital momentum [9]

$$\boldsymbol{M} = \frac{R^2}{8\pi k}\,\mathrm{Im}\left[\int_{r=R}\mathrm{d}^2\hat{\boldsymbol{r}}\ \ (\boldsymbol{E}^*{\times}\boldsymbol{E} + E_m^*(\boldsymbol{r}{\times}\nabla)E_m)\right]. \tag{7}$$

In particular, the existence of orbital AM expressed by the second term is interesting since polarized radiation by a source subjected to a magnetic field may be more intuitive to accept in view of the Faraday effect.

3. Environment of N dipoles with Zeeman shift

The Helmholtz Green's function for light scattering from N electric dipoles can be found in the literature [12–14]. Because of the point-like nature of the dipoles, it reduces to a $3N\times 3N$ complex-symmetric non-hermitian matrix. The magnetic response of the dipoles — due to the Zeeman splitting of their internal resonance — can be extracted in linear order so that the AM linear in the external field can be calculated numerically, given the N positions of the dipoles. The polarizability of a single dipole is given by

$$\boldsymbol{\alpha}(\omega,\boldsymbol{B}_0) = \alpha(0)\frac{\omega_0^2}{\omega_0^2 - (\omega{+}\omega_c\mathrm{i}\boldsymbol{\epsilon}\cdot\hat{\boldsymbol{B}}_0)^2 - \mathrm{i}\gamma\omega} \tag{8}$$

in terms of the radiative damping rate γ, the resonant frequency ω_0 and the cyclotron frequency $\omega_c = eB_0/(2mc_0)$. The second rank operator $\mathrm{i}\boldsymbol{\epsilon}\cdot\hat{\boldsymbol{B}}_0$ has three eigenvalues $0, \pm 1$ corresponding to 3 Zeeman levels that make the environment linearly magneto-birefringent. The external magnetic field $\boldsymbol{B}_0$ is assumed homogeneous across the environment, but this can easily be altered in future work, e.g., to describe an environment surrounding a magnetic dipole. The detuning parameter is defined as $\delta \equiv (\omega - \omega_0)/\gamma$. It is also useful to introduce the dimensionless material parameter $\mu = (12\pi/(\alpha(0)k^3))\times\omega_c/\omega_0$ that quantifies the magnetic birefringence induced by one dipole and typically small (for the $1S$–$2P$ transition in atomic hydrogen one can estimate $\mu \sim 7\cdot 10^{-5}$ Gauss^{-1}). The mean free path ℓ is another important length scale and follows from $1/\ell = nk\,\mathrm{Im}[\alpha(\omega)]$ if we neglect recurrent scattering, with $n = 3N/(4\pi a^3)$ the density of dipoles. Without a magnetic field, the polarizability can be written as $\alpha = -(3\pi/k^3)/(\delta + \mathrm{i}/2)$.

The leak of AM is calculated by performing the surface integral in (3) numerically for different realizations in which dipole positions are averaged over a sphere with a given radius a at homogeneous average density throughout the sphere. For more details we refer to [8]. It was explicitly checked that the surface integral did not depend on the choice of $R > a$ as required by the conservation of AM. Once verified, it is convenient to evaluate (3) in the far field $R \gg a$ where the fields simplify. Our code was also tested on flux conservation and obeys the optical theorem.

4. Numerical results

In this paper, we focus on numerical results obtained for different detunings δ and relatively large optical depths $\tau \equiv a/\ell$ in the hope of seeing major trends that can be extrapolated to even larger detunings. This regime becomes rapidly challenging since $\tau \sim \frac{9}{4}N/(ka \times \delta)^2$, so for $\delta \gg 1$ and $\tau \gg 1$ we need a large N. Typically, for $\delta = 2$, the best we have done so far, and $\tau = 5$, we already need $N = 2000$ in a sphere of 13 inverse wave numbers in radius ($ka = 13$). These numbers imply a value for the number of dipoles per optical volume $\eta \equiv 4\pi n/k^3 = 3N/(ka)^3 \sim 2.5$, i.e., the dipoles are largely located in each other's near field. Nevertheless, for a detuning $\delta = 2$, dipoles still scatter more or less independently because $k\ell \sim 2\delta^2/\eta = 3 > 1$ [15], but this is no longer true for $\eta = 6$. This implies that completely unknown effects, such as weak localization, may affect the radiative transfer of AM.

After an ideal average over all N dipole positions, the magnetic field is the only orientation left in the problem, and we expect that $\boldsymbol{M} = \kappa \hat{\boldsymbol{B}}_0$ with κ a real-valued scalar to be calculated that can have both signs. Following earlier works [4, 5, 8], we normalize the leakage of AM by the radiated amount of energy and introduce the dimensionless AM $\kappa\omega/P$, with P being the radiated amount of energy per second. This number is linear in the material parameter μ introduced earlier and can directly be related to the Hall angle of the Poynting vector in the far field of the sphere. Alternatively, the number quantifies the amount of leaked angular momentum expressed in $\hbar$, normalized per emitted photon.

In Figs. 2 and 3, we show the normalized AM leakage for an optically thin sphere as a function of detuning. The bars in all figures denote the typical support of the full probability distribution function (PDF) when calculating the torque for 1000 different realizations of the dipole positions. Except for the spin leakage rate, they are large, and all AM related to source and orbital momentum are genuine mesoscopic parameters. The optical depth $\tau = 1.9$ and average density $\eta = 0.3$ are kept constant, which means that both the number N of dipoles and the radius a of the sphere change as δ is varied. It is seen that the dimensionless AM depends significantly on detuning and changes sign near the resonance at $\delta = 0$. In this weakly scattering regime, single scattering is still dominant. For a thin layer in the far field of the source, we can derive a profile $\kappa\omega/P \sim \eta\,\mathrm{Im}[\alpha^2] \sim -\eta\delta/(\delta^2 + 1/4)^2$ independent of distance. This corresponds more or less to the observed profile of total AM in Figs. 2 and 3 that are nevertheless affected by higher-order scattering events. Both figures also show how total AM leakage splits up into either spin + orbital AM (Fig. 2) or torque on source + torque on the environment (Fig. 3). All are of the same order of magnitude but are not always of same sign. In particular, for this set of parameters, both decompositions have opposite signs.

The picture changes significantly in the multiple scattering regime. Figures 4 and 5 show the same normalized leakage of AM for optical depth $\tau = 4.6$

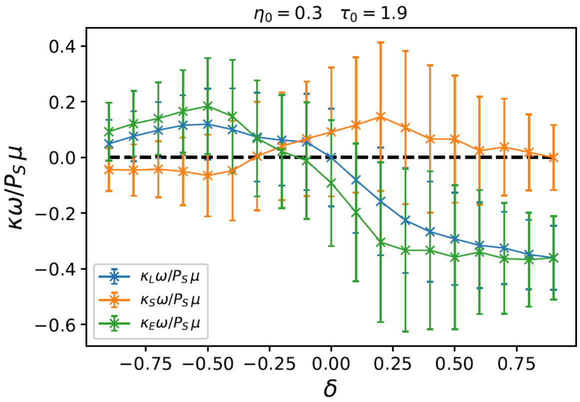

Fig. 2. Total normalized leakage rate of angular momentum (blue) as a function of detuning $\delta = (\omega - \omega_0)/\gamma$ from the dipole resonance, separated into torque on the source(orange) and torque on the environment (green). The optical depth is $\tau = 1.9$, and the dimensionless dipole density is $\eta = 0.3$.

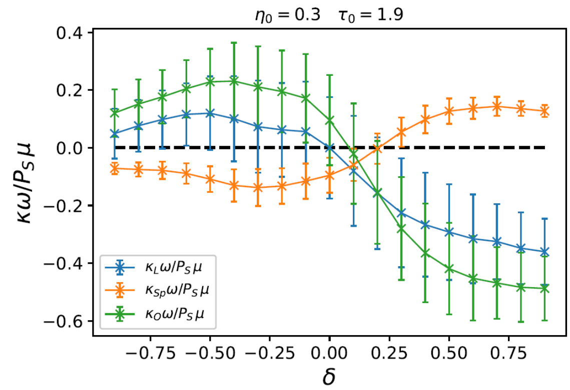

Fig. 3. As in previous figure with same fixed parameters $\tau = 1.9$ and $\eta = 0.3$, but this time, the total leakage has been split up into leakage of orbital angular momentum (green) and leakage of spin (orange).

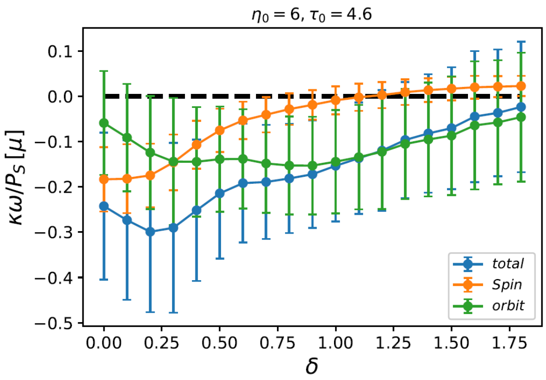

Fig. 4. Total leakage of optical angular momentum (blue) as a function of detuning. The orange and green curves represent spin and orbital momentum. The optical depth and average density are kept constant ($\tau = 4.6$ and $\eta = 6$). The calculations have been done only for $\delta > 0$, i.e., blue-shifted from the resonance.

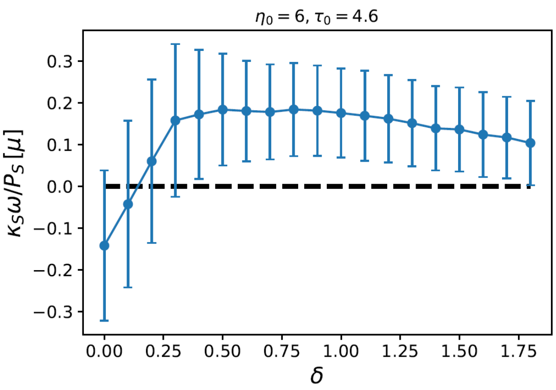

Fig. 5. As in Fig. 4, for clarity only the torque on the source has been shown. Bars denote the support of the full probability distribution (PDF) over 1000 realizations.

and dimensionless density as large as $\eta = 6$. Going to smaller values of η would require too large a value for N. The total leakage is now negative for all (positive) detunings, and spin leakage and orbital leakage have the same sign. Except for near resonance, it is dominated by leaks in orbital AM. In Fig. 5, we see that the torque on the source is mainly positive, but changes sign with detuning near resonance. This implies again that for most detunings, source and environment are subject to opposite torques. A normalized torque on the source around 0.1–0.2 is not much different than that found for $\eta = 0.3$ [8]. For low densities, this torque increased with η but seems to saturate for $\eta > 0.3$. The numbers for total leakage rate (-0.2 ± 0.1) are almost one order of magnitude less than what we found for $\eta = 0.03$ and $\tau \approx 2$ in [8], and one may speculate about the possibility of some process related to "weak localization" that reduces the transfer of angular momentum.

To get an order of magnitude for this effect, we consider a homogenous gas with atoms of mass $Z m_H$ in a sphere of size a. For an ideal gas at room temperature, the density is roughly $n_0 = 40\ \mathrm{mol/m^3}$. If we assume that all leaked angular momentum after a time interval Δt is transferred homogeneously to the mechanical momentum of the sphere, we can estimate that its angular velocity is

$$\Omega \left[\frac{\mathrm{rad}}{\mathrm{s}}\right] = 5.7 \frac{\kappa\omega}{P\mu}\,\frac{\mu}{Z}\,\frac{\left(\frac{P}{10\,\mathrm{W}}\right)\left(\frac{\lambda}{500\,\mathrm{nm}}\right)\left(\frac{\Delta t}{\mathrm{days}}\right)}{\left(\frac{a}{1\,\mathrm{cm}}\right)^5\left(\frac{n}{n_0}\right)}. \tag{9}$$

For $\kappa\omega/P\mu = 0.3$, $\mu/Z \sim 10^{-5}$, the angular rotation is typically of the order of 1 mrad/s after 100 days.

5. Conclusions

In this work, we have reported an exact numerical study of the radiation of electromagnetic angular momentum by a light source imbedded in a disordered and magneto-active environment described by resonant electric dipoles with Zeeman splitting that scatter light elastically. The angular momentum is directed along the magnetic field, and its transfer is directed radially outward from the source. It is, in general, composed of both spin transport and transport of orbital angular momentum. The first implies polarization of radiated light, the second is related to an energy flux circulating around the object and the magnetic field. Leakage of angular momentum has been quantified by a dimensionless parameter that is essentially the ratio of angular momentum leakage rate (with physical unit Joule) and the energy of the source emitted during one optical cycle. The number can be seen to be equal to the angular momentum, expressed in $\hbar$, transferred per emitted photon to the source and environment. It is proportional to the product of a pure material parameter μ associated with the magneto-scattering of the dipoles and the numbers that can be found in the figures that result from scattering. By conservation of angular momentum, this transfer gives rise to torques on both the emitting source and the scattering environment. All parameters are seen to be of the same order, can have mutually opposite signs, and depend on the detuning from the resonance.

The regime of Thompson scattering is interesting for astrophysical applications and is a major scattering mechanism in our Sun. It corresponds to large detunings, where the phase shift between the incident and scattered field becomes negligible. The out-of-phase response seems essential for the leakage to exist, even for large optical depths. The amount of multiple scattering, quantified by optical depth, certainly affects the radiative transfer of angular momentum, but it is difficult at this point to deduce general trends. Our simulations are clearly in need of a radiative transport theory for angular momentum in magnetic fields, which, to our knowledge, does not exist. From our simulations, we suspect that different parts of the environment undergo different torques. We also expect that as the optical thickness of the environment increases, the precise nature of the source becomes of less importance, quite opposite to what was found for a homogeneous environment.

Indeed, this picture may possibly apply to stellar atmospheres, globally exposed to the magnetic dipole fields of their nuclei. Although all ingredients are present for leakage of angular momentum to exist, lots of extra complications, such as broadband radiation, Doppler broadening, etc., make quantitative predictions difficult.

Acknowledgments

This work was funded by the Agence Nationale de la Recherche (Grant No. ANR-20-CE30-0003 LOLITOP).

References

[1] D. Kleppner, *Phys. Rev. Lett.* **47**, 233 (1981).

[2] E. Yablonovitch, *Phys. Rev. Lett.* **58**, 2059 (1987).

[3] R. El-Dardiry, S. Faenz, A. Lagendijk, *Phys. Rev. A* **83**, 031801 (2011).

[4] B.A. van Tiggelen, G.L.J.A. Rikken, *Phys. Rev. Lett.* **125**, 133901 (2020).

[5] B.A. van Tiggelen, *Opt. Lett.* **48**, 41 (2023).

[6] E.M. Purcell, *Phys. Rev.* **69**, 681 (1946).

[7] D.G. Baranov, R.S. Savelev, S.V.Li, A.E. Krasnok, A. Alu, *Laser Photon. Rev.* **11**, 1600268 (2017).

[8] R. Le Fournis, B.A. van Tiggelen, *Phys. Rev. A* **108**, 053713 (2023).

[9] C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, *Photons and Atoms*, Wiley, 1989.

[10] J.D. Jackson, *Classical Electrodynamics*, Wiley, Hoboken (NJ) 1999.

[11] L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields*, Pergamon, 1975, p. 193.

[12] M. Rusek, A. Orlowski, J. Mostowski, *Phys. Rev. E* **56**, 4892 (1997).

[13] F.A. Pinheiro, M. Rusek, A. Orlowski, B.A. van Tiggelen, *Phys. Rev. E* **69**, 026605 (2004).

[14] O. Leseur, R. Pierrat, J.J. Saenz, R. Carminati, *Phys. Rev. A* **90**, 053827 (2014).

[15] A. Lagendijk, B.A. van Tiggelen, *Phys. Rep.* **270**, 143 (1996).