Dedicated to Professor Iwo Białynicki-Birula on His 90th Birthday

Fractality of Certain Quantum States

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Fractal structures appearing in solutions of certain quantum problems are investigated. We prove the previously announced results concerning the existence and properties of fractal states for the Schrödinger equation in the infinite one-dimensional well. In particular, we show that for this problem, there exist solutions in the form of *fractal quantum carpets*: the probability density $P(x,t)$ forms a fractal surface with dimension D_{xy}, while its cross-sections $P_t(x)$ and $P_x(t)$ typically form fractal graphs with dimensions D_x and D_t respectively, where $D_{xy} = 2 + D_x/2$ and $D_t = 1 + D_x/2$ (almost everywhere).

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1. Introduction

Fractals are sets and measures of non-integer dimension [1, 2]. They are good models of phenomena and objects in various areas of science. Their ubiquity in dynamical systems theory as attractors, repellers, and attractor boundaries is well-known [2, 3]. They are often connected with non-equilibrium problems of growth [4] and transport [5, 6]. Fractal properties of hydrodynamic modes have been shown to be connected with transport coefficients [7, 8]. Fractal dimensions are used in many nonlinear time series analysis methods [9, 10].

Fractals have also been found in quantum mechanics [11–14]. For instance, quantum models related to the problem of chaotic scattering often reveal fractal structures [15–17] relevant for quantum transport [18]. Fractal structures play a prominent role in studies of the quantum dynamics of a reduced density operator [19]. Spectroscopic characterization of the electronic wave function inside a confined structure with fractal geometry was discussed in [20]. Quantum field theories in fractal spacetimes were also analyzed [21, 22], and fractal structures were reported in models of quantum gravity [23, 24]. Fractional calculus was found useful to describe the dynamics of quantum particles [25], while a Bohmian approach to quantum fractals was presented in [26].

It was also shown that the Schrödinger equation for the simplest non-chaotic potentials admits fractal solutions [27]. The resulting probability distributions $P(x,t)$ as functions of space and time, called *quantum carpets* [28–31], reveal fractal features [27, 32, 33]. In this paper, we have two objectives. One is to present the rigorous proofs of fractality of quantum states reported in [32]. The other is to illustrate a convenient method of calculating the dimensions of graphs of continuous functions introduced by Claude Tricot [34].

2. Methods

2.1. Box-counting dimension

In this section, we recall several equivalent definitions of box-counting dimension, state a criterion for finding the dimension of a continuous function of one variable, and prove a connection between the dimension of the graph of a function of n variables and the dimensions of its sections. All of this is known, perhaps with the exception of Theorem 2, which might be new. We concentrate on the theory of box-counting dimension for graphs of continuous functions of one variable. More general theory and a deeper presentation can be found, for instance, in [2, 34, 35].

Let $A \subset \mathbb{R}^n$ be bounded. Consider a grid of n-dimensional boxes of side δ

$$\big[m_1\delta, (m_1{+}1)\delta\big] \times \cdots \times \big[m_n\delta, (m_n{+}1)\delta\big]. \tag{1}$$

Let $N(\delta)$ be the number of these boxes covering the set A. It is always finite because A is bounded.

Definition 1. Box-counting dimension of the set A is the limit

$$\dim_B(A) := \lim_{\delta\to 0} \frac{\ln\big[N(\delta)\big]}{\ln\big[1/\delta\big]}. \tag{1}$$

If the limit does not exist, one considers upper and lower box-counting dimensions

$$\overline{\dim}_B(A) := \limsup_{\delta\to 0} \frac{\ln\big[N(\delta)\big]}{\ln\big[1/\delta\big]}, \tag{3}$$

$$\underline{\dim}_B(A) := \liminf_{\delta\to 0} \frac{\ln\big[N(\delta)\big]}{\ln\big[1/\delta\big]}, \tag{4}$$

which always exist and satisfy

$$\overline{\dim}_B(A) \geq \underline{\dim}_B(A). \tag{5}$$

The box-counting dimension exists if the upper and lower box-counting dimensions are equal.

Several equivalent definitions are in use (see [1, 2, 34, 35] for a review). The most convenient definition to study the fractal properties of graphs of continuous functions is given in terms of δ-variations [34]. It is essentially a variant of the Bouligand definition [36]. We shall restrict our attention to dimensions of curves being subsets of a plane.

Let $K_\delta(x)$ be a closed ball $\{y \in \mathbb{R}^2 : |x-y| \leq \delta\}$.

Definition 2. *Minkowski sausage* or *δ-parallel body* of $A \subset \mathbb{R}^2$ is

$$A_\delta := \bigcup_{x\in A} K_\delta(x) = \{y \in \mathbb{R}^2 : \exists\, x \in A, |x{-}y| \leq \delta\}. \tag{6}$$

Thus the Minkowski sausage of A is the set of all the points located within δ of A.

Proposition 1. The box-counting dimension of a set $A \subset \mathbb{R}^2$ satisfies

$$\dim_B(A) = \lim_{\delta\to 0}\left(2 - \frac{\ln\big[V(A_\delta)\big]}{\ln[\delta]}\right), \tag{7}$$

where $V(\delta) = \mathrm{vol}^2(A_\delta)$ is the area of the Minkowski sausage of A.

Proof. Every square from the δ-grid containing $x \in A$ is included in $K_{\sqrt{2}\delta}(x)$. On the other hand, every closed ball of radius $\sqrt{2}\delta$ can be covered by at most 16 squares from the grid. Therefore,

$$\delta^2 N(\varepsilon) \leq V\big(A_{\sqrt{2}\delta}\big) \leq 16\delta^2 N(\varepsilon). \tag{8}$$

□

Consider a continuous function on a closed interval $f : [a, b] \to \mathbb{R}$. Its graph is a curve in the plane. To find its box-counting dimension, estimate the number of boxes $N(\delta)$ intersecting the graph. Choose column $\{(x, y) : x \in [n\delta, (n+1)\delta]\}$. Since the curve is continuous, the number of the boxes in this column intersecting the graph of f is at least

$$\frac{1}{\delta}\left[\sup_{x\in[n\delta,(n+1)\delta]} f(x) - \inf_{x\in[n\delta,(n+1)\delta]} f(x)\right] \tag{9}$$

and no more than the same plus 2. If f was a record of a signal, then the difference between the maximum and minimum value of f on the given interval quantifies how the signal oscillates on this interval. That's why it is called *δ-oscillation.*

Definition 3. δ-oscillation of f at x is

$$\mathrm{osc}_\delta(x)(f) := \sup_{|y-x|\leq\delta} f(y) - \inf_{|y-x|\leq\delta} f(y) = \sup\big\{|f(y){-}f(z)| : \, y, z \in [a, b] \cap [x{-}\delta, x{+}\delta]\big\}. \tag{10}$$

We will skip (f) if it is clear from the context which function we consider.

From (9) we obtain the following estimate on the total number of boxes covering the graph of f,

$$\sum_{m=1}^{M} \frac{\mathrm{osc}_{\delta/2}(x_m)}{\delta} \leq N(\delta) \leq 2M + \sum_{m=1}^{M} \frac{\mathrm{osc}_{\delta/2}(x_m)}{\delta}, \tag{11}$$

where $x_m = a + (m - \frac{1}{2})\delta$ is the middle of the m-th column from the cover of the graph and $M = \lceil \frac{b-a}{\delta} \rceil$ is the number of columns in the cover ($\lceil x \rceil$ stands for the smallest integer greater or equal to x). Thus,

$$N(\delta) \approx M\, \overline{\mathrm{osc}}_{\delta/2}/\delta. \tag{12}$$

If the graph of f has the box-counting dimension D, $N(\delta)$ scales as δ^{-D}. This implies the following scaling of the oscillations

$$\overline{\mathrm{osc}}_{\delta/2} \approx N(\delta)\,\delta/M \propto \delta^{2-D}. \tag{13}$$

We have thus suggested a connection between the box-counting dimension of the graph and the scaling exponent of the average oscillation of the function f.

Definition 4. δ-variation of function f is

$$\mathrm{Var}_\delta(f) := \int_a^b \mathrm{d}x\; \mathrm{osc}_\delta(x)(f) =: (b{-}a)\, \overline{\mathrm{osc}}_\delta(f). \tag{14}$$

Geometrically, variation is the area of the set scanned by the graph of f moved horizontally $\pm\delta$ and truncated at $x = a$ and $x = b$, thus, it is a kind of Minkowski sausage constructed with horizontal intervals of length 2δ. This observation leads to a convenient technique for calculating dimensions.

Theorem 1. Let $f(x)$ be a non-constant continuous function on $[a,b]$, then

$$\dim_B \mathrm{graph} f = \lim_{\delta\to 0}\left(2-\frac{\ln\left[\mathrm{Var}_\delta(f)\right]}{\ln[\delta]}\right). \quad (15)$$

The proof consists of showing equivalence of $\mathrm{Var}_\delta(f)$ with the Minkowski sausage and follows from inequality ([34], p. 130–132, 148–149)

$$\mathrm{Var}_\delta(f) \le V(A_\delta) \le c\,\mathrm{Var}_\delta(f), \quad (16)$$

where

$$A = \mathrm{graph} f,$$
$$c = c_1 + c_2/s,$$
$$s = \Big[\sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x)\Big]. \quad (17)$$

This is where the assumption of non-constancy of f comes in. Derivation of (16) is not difficult but rather lengthy and will be omitted.

This theorem is the main tool to prove Theorem 3 in Sect. 3. In order to find the dimensions, we will look for estimates of δ-variation. They will usually take the following form:

Proposition 2

1. $\mathrm{osc}_\delta(x) f(x) \le c\,\delta^{2-s} \Rightarrow \dim_B \mathrm{graph}\, f \le s$.
2. $W := \int_a^b \mathrm{d}x\, \left|f(x+\delta) - f(x-\delta)\right| \ge c\,\delta^{2-s} \Rightarrow \dim_B \mathrm{graph}\, f \ge s$.

Proof.

1. $\mathrm{Var}_\delta f = \int_a^b \mathrm{d}x\, \mathrm{osc}_\delta(x)(f) \le (b-a)c\,\delta^{2-s}$.
2. $\mathrm{osc}_{2\delta}(x) f \ge \left|f(x+\delta) - (x-\delta)\right| \Rightarrow \mathrm{Var}_\delta f \ge (b-a)c\,(\delta/2)^{2-s}$. □

To prove the last point of Theorem 3, we need to know what is the dimension of the graph of $f : \mathbb{R}^n \to \mathbb{R}$, given all the dimensions of its one-variable restrictions.

Theorem 2. Let $f \in \mathcal{C}^0([a_1,b_1]\times\cdots\times[a_n,b_n])$. For every point $x = (x^1,\ldots,x^n) \in [a_1,b_1]\times\cdots\times[a_n,b_n]$ define $\tilde{x}^i := (x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n)$. Then

$$f_i[\tilde{x}^i_0](x^i) := f(x^1_0,\ldots,x^{i-1}_0,x^i,x^{i+1}_0,\ldots,x^n_0) \quad (18)$$

is a restriction of f to a line parallel to i-th axis going through x_0 and $f_i[\tilde{x}^i_0] \in \mathcal{C}^0([a_i,b_i])$.

(1) If $\forall x : \mathrm{osc}_\delta f_i[\tilde{x}^i] \le c_i\delta^{H_i}$, then

$$\dim_B \mathrm{graph}\, f(x^1,\ldots,x^n) \le n+1-\min\{H_1,\ldots,H_n\}. \quad (19)$$

(2) If $\mathrm{Var}_\delta f_i[\tilde{x}^i_0] \ge c_i\,\delta^{H_i}$ for a dense set $\tilde{x}^i_0 \in A \subset A = [a_1,b_1]\times\ldots\times[a_{i-1},b_{i-1}]\times[a_{i+1},b_{i+1}]\times\ldots\times[a_n,b_n]$, then

$$\dim_B \mathrm{graph} f(x^1,\ldots,x^n) \ge n+1-\min\{H_1,\ldots,H_n\}. \quad (20)$$

(3) If all of the above conditions are satisfied, then

$$\dim_B \mathrm{graph} f(x^1,\ldots,x^n) = n+1-\min\{H_1,\ldots,H_n\} = n-1+\max\{s_1,\ldots,s_n\}, \quad (21)$$

where $s_i = \sup_{\tilde{x}^i} \dim_B \mathrm{graph}\, f_i[\tilde{x}^i](x^i)$.

In other words, the strongest oscillations along any direction determine the box-counting dimension of the whole $n+1$-dimensional graph.

Proof. We will show the theorem for $n = 2$ for notational simplicity. Generalization to arbitrary n is immediate. Let $f : [a_1,b_1]\times[a_2,b_2] \to \mathbb{R}$. Divide the domain into squares $X_i \times Y_j$ of side δ. This gives rise to K columns A_{ij} of δ-grid in $\mathbb{R}^3$, $1 \le \frac{K\,\delta^2}{(b_1-a_1)(b_2-a_2)} \le 2$.

(1). The number of δ-cubes having a common point with the graph of f in column A_{ij} is not greater than $\frac{1}{\delta}(\sup_{A_{ij}} f - \inf_{A_{ij}} f)+2$. But

$$\begin{aligned}\left|f(x_1,y_1) - f(x_2,y_2)\right| &= \\ &\left|f(x_1,y_1)-f(x_1,y_2) + f(x_1,y_2)-f(x_2,y_2)\right| \\ &\le \left|f(x_1,y_1)-f(x_1,y_2)\right| + \left|f(x_1,y_2)-f(x_2,y_2)\right|.\end{aligned} \quad (22)$$

Therefore

$$\begin{aligned}\sup_{A_{ij}} f - \inf_{A_{ij}} f &= \sup_{(x_1,y_1),(x_2,y_2)\in A_{ij}} \left|f(x_1,y_1)-f(x_2,y_2)\right| \\ &\le \sup_{x\in X_i}\sup_{y\in Y_j} f(x,y) + \sup_{y\in Y_j}\sup_{x\in X_i} f(x,y) \\ &\le \sup_{x\in X_i} \mathrm{osc}_{\delta/2}\, f_1[x] + \sup_{y\in Y_j} \mathrm{osc}_{\delta/2}\, f_2[y] \\ &\le c\,\delta^{\min\{H_1,H_2\}}.\end{aligned} \quad (23)$$

Thus

$$\begin{aligned}\dim_B \mathrm{graph}\, f(x^1,x^2) &\le \lim_{\delta\to 0} \frac{\ln[Kc\,\delta^{\min\{H_1,H_2\}}/\delta]}{\ln[1/\delta]} \\ &\le 3-\min\{H_1,H_2\}.\end{aligned} \quad (24)$$

(2). Set $x \in X_i$. From (8) and (16) it follows that the number $N_i(\delta)$ of δ-cubes in columns A_{ij} covering the graph of $f_2[x](y)$ and the variation of f_2 satisfy

$$\mathrm{Var}_\delta f_2[x] \le c\,\delta^2 N_i(\delta). \quad (25)$$

Thus

$$N_i(\delta) \ge c \sup_{x\in X_i} \mathrm{Var}_\delta f_2[x]/\delta^2 \ge c\,\delta^{H_2-2}. \quad (26)$$

Therefore, the number $N(\delta)$ of boxes covering the whole graph of f satisfies

$$N(\delta) \ge c \sum_{i=1}^{M} \sup_{x\in X_i} \mathrm{Var}_\delta f_2[x]/\delta^2 \ge c_1\delta^{H_2-3}. \quad (27)$$

The same can be repeated for any direction, thus

$$N(\delta) \ge c_2\,\delta^{\min\{H_1,H_2\}-3}. \quad (28)$$

(3). An immediate corollary.

Generalization to arbitrary n is achieved by observing that $K\delta^n \approx \mathrm{const}$. □

Another definition, which has some convenient technical properties, is the Hausdorff dimension [37–39], however, it is often too difficult to calculate. For instance, as far as we know, there is still no proof that the Hausdorff dimension of the Weierstrass function is equal to its box-counting dimension. Thus in practice, one usually uses the (upper) box-counting dimension. This is also our present approach. It is often assumed that the box-counting dimension and the Hausdorff dimension are equal. A general characterization of situations when this conjecture really holds is also lacking.

2.2. Fractal functions

One of the oldest fractals is a graph of the Weierstrass function [40, 41]

$$W(x) = \sum\nolimits_{n=0}^{\infty} a^n \cos(b^n x\pi), \tag{29}$$

introduced as an example of an everywhere continuous, nowhere differentiable function by Karl Weierstrass around 1872. The maximum range of parameters, for which the above series has required properties was found by Godfrey Harold Hardy in 1916 [42], who also showed that

$$\sup\left\{|f(x){-}f(y)| : |x{-}y| \le \delta\right\} \sim \delta^H, \tag{30}$$

where $H = \frac{\ln(1/a)}{\ln(b)}$. From this it easily follows (see below) that the box-counting dimension of the graph of the Weierstrass function $W(x)$ is

$$D_W = 2{+}H = 2{+}\frac{\ln(a)}{\ln(b)} = 2{-}\left|\frac{\ln(a)}{\ln(b)}\right|. \tag{31}$$

Functions whose graphs have non-integer box-counting dimension are called *fractal functions*. Even though the box-counting dimension of the Weierstrass function is easy to calculate [34], the proof that its Hausdorff dimension has the same value is still lacking, as far as we know. Lower bounds on the Hausdorff dimension of the graph were found by Mauldin [43, 44]. Graphs of random Weierstrass functions were shown to have the same Hausdorff and box-counting dimensions for almost every distribution of phases [45].

3. Results

The construction of the Weierstrass function, (29), can easily be realized in quantum mechanics. Consider solutions of the Schrödinger equation

$$\mathrm{i}\,\partial_t\,\Psi(x,t) = -\nabla^2\,\Psi(x,t) \tag{32}$$

for a particle in the one-dimensional infinite potential well. The general solutions satisfying the boundary conditions $\Psi(0,t) = 0 = \Psi(\pi,t)$ have the form

$$\Psi(x,t) = \sum\nolimits_{n=1}^{\infty} a_n \sin(nx)\,\mathrm{e}^{-\mathrm{i}\,n^2 t}, \tag{33}$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} \mathrm{d}x\ \sin(nx)\,\Psi(x,0). \tag{34}$$

Weierstrass quantum fractals are wave functions of the form

$$\Psi_M(x,t) = N_M \sum\nolimits_{n=0}^{M} q^{n(s-2)}\sin(q^n x)\,\mathrm{e}^{-\mathrm{i}\,q^{2n}t}, \tag{35}$$

where $q = 2, 3, \ldots,\ s \in (0,2)$.

In the physically interesting case of finite M, the wave function Ψ_M is a solution of the Schrödinger equation. The limiting case

$$\Psi(x,t) := \lim_{M\to\infty} \Psi_M(x,t) = N\sum\nolimits_{n=0}^{\infty} q^{n(s-2)}\sin(q^n x)\,\mathrm{e}^{-\mathrm{i}\,q^{2n}t}, \tag{36}$$

with the normalization constant $N{=}\sqrt{\frac{2}{\pi}}\sqrt{1{-}q^{2(s-2)}}$, is continuous but nowhere differentiable. It is a weak solution of the Schrödinger equation. Note that (36) converges for $(|q^{s-2}| < 1 \equiv s < 2$. Since the probability density of wave function (36) shows fractal features for $s > 0$ (see below), the interesting range of s is $(0,2)$.

The main results announced in [32], which we prove here, are that not only the real part of the wave function $\Psi(x,t)$, but also the physically important probability density $P(x,t) := |\,\Psi(x,t)|^2$ exhibit fractal nature. This is not obvious, because $|\,\Psi(x,t)|^2$ is the sum of squares of the real and imaginary parts having usually equal dimensions. One can easily show that the dimension of the graph of a sum of functions whose graphs have the same dimensions D can be anything[†] from 1 to D.

Our main results are given by Theorem 3.

Theorem 3. Let $P(x,t)$ denote the probability density of a Weierstrass-like wave function (36). Then

1. at the initial time $t = 0$, the probability density $P_0(x) = P(x,0)$ forms a fractal graph in the space variable (i.e., space fractal) of dimension $D_x = \max\{s,1\}$;
2. the dimension D_x of graph of $P_t(x) = P(x, t = \mathrm{const})$ does not change in time;
3. for almost every x inside the well, the probability density, $P_x(t) = P(x = \mathrm{const}, t)$, forms a fractal graph in the time variable (i.e., time fractal) of dimension $D_t(x) = D_t := 1{+}s/2$;
4. for a discrete, dense set of points x_d, $P_{x_d}(t) = P(x_d,t)$ is smooth, and thus $D_t(x_d) = 1$;
5. for even q, the average velocity $\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t}(t)$ is fractal with the dimension of its graph equal to $D_v = \max\{(1+s)/2, 1\}$;
6. the surface $P(x,t)$ has dimension $D_{xy} = 2{+}s/2$.

[†1] Let f_1 and f_2 be functions with graphs having dimensions, respectively, $1 \le D_1 < D_2 \le 2$. Let $g_1 = f_1 + f_2$, $g_2 = f_1 - f_2$. Then the box-counting dimension of both the graph of g_1 and g_2 is D_2, but the dimension of the graph of their sum $g_1 + g_2 = 2f_1$ is $D_1 \in [1, D_2]$.

The physical meaning of Theorem 3 has been discussed in [32]. Here we only emphasize that to generate a fractal wave function with exact mathematically rigorous fractal features with infinite scaling properties, infinite energy is required. However, even a few terms in the series defining the function (36) can lead to physically interesting effects.

Our proof of Theorem 3 is based on the power-law behavior of the average δ-oscillation of the infinite double sum present in $P(x,t) = |\Psi(x,t)|^2$ (see (122) in Appendix). Some fundamental concepts and facts used in the proof are given in Sect. 2.1. Calculations of probability density and average velocity are provided in the Appendix. Positive real constants are denoted by $c, c_1, c_2, \ldots$

4. Proof of Theorem 3

1. *At the initial time $t = 0$, the probability density, $P_0(x) = P(x,0)$, forms a fractal graph in the space variable (i.e.,* space fractal*) of dimension $D_x = \max\{s,1\}$.*
2. *The dimension D_x of graph of $P_t(x) = P(x, t = const)$ does not change in time.*

We will show that for every fixed t, the graph of the probability density $|\Psi|^2$ (see (122) in Appendix) as a function of x has the box-counting dimension s.

(a) Fix t. Let

$$P_n(x) := \sum_{k=0}^{n} q^{k(s-2)} \sum_{l=0}^{k} \sin(q^l x)\, \sin(q^{k-l} x) \times \cos\left[\left(q^{2l} - q^{2(k-l)}\right)t\right], \tag{37}$$

$q = 2, 3, \ldots$ It is a smooth function whose derivative at every point satisfies

$$|P_n'(x)| \le 2 \sum_{k=0}^{n} q^{k(s-2)} \sum_{l=0}^{k} q^l \left|\cos\left(q^l x\right) \sin\left(q^{k-l} x\right)\right| \le 2 \sum_{k=0}^{n} \tfrac{q^{k+1}}{q-1}\, q^{k(s-2)} \le d_1(s,q)\, q^{n(s-1)}, \tag{38}$$

where

$$d_1(s,q) = \frac{2q^s}{(q-1)(q^{s-1}-1)}. \tag{39}$$

Let $\delta = q^{-n}$. Then

$$\mathrm{osc}_\delta(x) P_n \le 2\delta \sup_{x\in[0,\pi]} |P_n'(x)| \le 2 d_1(s,q) \delta^{2-s}. \tag{40}$$

On the other hand, for

$$R_n(x) := P(x) - P_n(x) = \sum_{k=n+1}^{\infty} q^{k(s-2)} \times \sum_{l=0}^{k} \sin(q^l x)\, \sin(q^{k-l} x)\, \cos\left[(q^{2l} - q^{2(k-l)})t\right], \tag{41}$$

we have

$$\mathrm{osc}_\delta(x) R_n \le 2 \sum_{k=n+1}^{\infty} q^{k(s-2)} (k+1) \le \frac{4q^{(n+1)(s-2)} n}{(1-q^{s-2})^2}. \tag{42}$$

Polynomial growth is slower than exponential, therefore for arbitrarily small ε there is some M such that $\forall n > M : n < (q^\varepsilon)^n$. This leads to the following estimate of the oscillation of R_n,

$$\mathrm{osc}_\delta(x) R_n \le d_2(s,q)\, \delta^{2-s-\varepsilon}, \tag{43}$$

where

$$d_2(s,q) = \frac{4q^{s-2}}{(1-q^{s-2})^2}. \tag{44}$$

Thus for all x and $\delta = q^{-n}$, where $\frac{\ln(n)}{n} < \varepsilon \ln(q)$, we have

$$\mathrm{osc}_\delta(x) P \le \mathrm{osc}_\delta(x) P_n + \mathrm{osc}_\delta(x) R_n \le \left(2d_1 + d_2\right) \delta^{2-s-\varepsilon}. \tag{45}$$

From Proposition 2 it follows that $\dim_B \mathrm{graph}\, P_t(x) \le 2 - (2-s-\varepsilon) = s+\varepsilon \to_{\varepsilon\to 0} s$.

(b) Fix t. Let $f(x) = P(x,t)$. We want to show that

$$W := \int_a^b \mathrm{d}x\, \left|f(x+\delta) - f(x-\delta)\right| \ge c\, \delta^{2-s}. \tag{46}$$

Take $a=0$, $b=\pi$. Notice that (we skip the normalization constant)

$$W = \int_0^\pi \mathrm{d}x\, \left|f(x+\delta) - f(x-\delta)\right| = \int_0^\pi \mathrm{d}x\, \left| \sum_{k=0}^{\infty} q^{k(s-2)} \sum_{l=0}^{k} \left\{ \sin[q^l(x+\delta)] \sin[q^{k-l}(x+\delta)] - \sin[q^l(x-\delta)] \sin[q^{k-l}(x-\delta)] \right\} a_{kl} \right| = \int_0^\pi \mathrm{d}x\, \left| \sum_{k=0}^{\infty} q^{k(s-2)} \sum_{l=0}^{k} \left\{ \cos(q^l x) \sin(q^{k-l} x) \sin(q^l \delta) \cos(q^{k-l}\delta) \right\} a_{kl} \right|, \tag{47}$$

where $a_{kl} = \cos[(q^{2k} - q^{2l})t]$. Take $|h(x)| \le 1$. Observe that

$$\int_a^b \mathrm{d}x\, \left|\sum_i f_i(x)\right| \ge \int_a^b \mathrm{d}x\, |h(x)| \left|\sum_i f_i(x)\right| \ge \left|\int_a^b \mathrm{d}x\, \sum_i h(x) f_i(x)\right| \ge \left|\int_a^b \mathrm{d}x\, h(x) f_k(x)\right| - \sum_{i\ne k} \left|\int_a^b \mathrm{d}x\, h(x) f_i(x)\right|. \tag{48}$$

One can interchange the order of summation and integration because $f(x)$ is absolutely convergent. Let us take $\delta{=}q^{-N}$, $h(x) = \sin(q^m x)\cos(q^n x)$. After substitution in (47) using (48) we obtain

$$W \geq \left|\sum_{k=0}^{\infty} q^{k(s-2)} \sum_{l=0}^{k} \sin(q^{l-N})\sin(q^{k-l-N}) \int_0^{\pi} \mathrm{d}x\ \sin(q^l x)\cos(q^{k-l}x)\cos(q^m x)\cos(q^n x)\, a_{kl}\right| =$$

$$\frac{\pi}{4} q^{(m+n)(s-2)} \left|\cos(q^{m-N})\sin(q^{n-N})\cos\left[(q^{2(m+n)}-q^{2m})t\right]\right| =: \frac{\pi}{4}\widetilde{W}. \tag{49}$$

We will now prove that $\exists c : \widetilde{W} = q^{(m+n)(s-2)} \times |\cos(q^{m-N})\sin(q^{n-N})\cos[(q^{2(m+n)} - q^{2m})t]| \geq cq^{N(s-2)}$, for arbitrary real t. We will take advantage of the fact that q is an integer.

Let $N = m+n$. Then $\widetilde{W} = q^{N(s-2)} \times |\cos(q^{m-N})\sin(q^{-m})\cos[(q^{2N}-q^{2m})t]|$. It is enough to consider $t \in [0,\pi]$.

Let us write t/π in q

$$\frac{t}{\pi} = \frac{a_1}{q} + \frac{a_2}{q^2} + \frac{a_3}{q^3} + \cdots = \sum_{k=1}^{\infty} \frac{a_k}{q^k}, \tag{50}$$

where $a_k \in \{0,1,\ldots,q-1\}$, so that t/π can be written as

$$\frac{t}{\pi} = 0.a_1 a_2 \ldots a_K \big(a_{K+1}\ldots a_{K+T}\big). \tag{51}$$

Therefore,

$$\cos\left[(q^{2N}-q^{2m})t\right] = \cos\left[\pi(q^{2N-1}a_1 + q^{2N-2}a_2 + \cdots + a_{2N} + q^{-1}a_{2N+1} + \cdots + q^{2m-1}a_1 + q^{2m-2}a_2 + \cdots + a_{2m} + q^{-1}a_{2m+1} + \ldots)\right] = \cos\left[\pi\left(\tfrac{a_{2N+1}-a_{2m+1}}{q} + \tfrac{a_{2N+2}-a_{2m+2}}{q^2} + \ldots\right)\right]. \tag{52}$$

If we could only choose m so that the first two terms in this series cancel out, we would have a lower estimate on the cosine, because, in this case,

$$\left|\frac{a_{2N+3}-a_{2m+3}}{q^3} + \ldots\right| \leq (q-1)\left(\frac{1}{q^3} + \frac{1}{q^4} + \ldots\right) = \frac{1}{q^2}. \tag{53}$$

Thus,

$$\cos\left[\left(q^{2N}-q^{2m}\right)t\right] \geq \cos\left(\frac{\pi}{q^2}\right) \geq \cos\left(\frac{\pi}{4}\right). \tag{54}$$

Let A be the set of all the two-element sequences with elements from the set $\{0,1,\ldots,q{-}1\}$. Thus

$$A = \big\{\{0,0\},\{0,1\},\ldots,\{0,q{-}1\},\{1,0\},\ldots, \ldots,\{q{-}1,q{-}1\}\big\}, \tag{55}$$

and we write $A_{k,l} := \{k,l\}$, $k,l \in \{0,1,\ldots,q-1\}$. Consider all the pairs of consecutive q-digits of t/π of the form

$$\big\{a_{2m+1}, a_{2m+2}\big\}, \tag{56}$$

i.e., $\{a_1,a_2\},\{a_3,a_4\}$ etc. Every such pair is equal to some $A_{k,l}$. Let $N_{k,l}$ be the first such m, for which

$$A_{k,l} = \big\{a_{2m+1}, a_{2m+2}\big\}. \tag{57}$$

If $A_{k,l}$ for given k,l doesn't appear in the sequence of all the pairs (56), we set $N_{k,l} = 0$. Let

$$M = \sup_{k,l} N_{k,l}. \tag{58}$$

Thus if $n{>}M$, the sequence $\{a_{2n+1}, a_{2n+2}\}$ has appeared at least once among the pairs $\{a_1,a_2\},\{a_3,a_4\},\ldots,\{a_{2M+1},a_{2M+2}\}$. Therefore, for every $N{>}M$ we can find such an $m \in 1,2,\ldots,M$ that

$$\left|\cos\left[(q^{2N}-q^{2m})\frac{t}{\pi}\pi\right]\right| \geq \cos\left(\frac{\pi}{q^2}\right) \geq \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}. \tag{59}$$

Also

$$\sin(q^{-m}) \geq \sin(q^{-M}),$$

$$\cos(q^{m-N}) \geq \cos(q^{M-N}) \geq \cos(1), \tag{60}$$

which leads to

$$\widetilde{W} \geq \frac{\sqrt{2}}{2} q^{N(s-2)}\sin(q^{-M})\cos(1) = \mathrm{const}\, q^{N(s-2)}. \tag{61}$$

We have thus shown that for every t, for natural q, and for $\delta = q^{-N}$

$$W \geq \mathrm{const}\,\delta^{2-s}, \tag{62}$$

therefore (Proposition 2)

$$\dim_B \mathrm{graph}\, P_t(x) \geq 2 - (2{-}s) = s. \tag{63}$$

3. *For almost every* x, $D_t(x) = \dim_B \mathrm{graph} P_x(t) = D_t := 1{+}s/2$.

We will use the form (123) given in Appendix of the probability density. It is enough to analyze the dimension of

$$\widetilde{P}(t) := \sum_{c=1}^{\infty} q^{2c(s-2)}\sin(q^c x)\sum_{d=1}^{c} q^{-d(s-2)} \times \sin(q^{c-d}x)\cos[(q^{2c}-q^{2(c-d)})t]. \tag{64}$$

(a) Let

$$P_n(t) := \sum_{c=1}^{n} q^{2c(s-2)}\sin(q^c x)\sum_{d=1}^{c} q^{-d(s-2)} \times \sin(q^{c-d}x)\cos[(q^{2c}-q^{2(c-d)})t]. \tag{65}$$

Then

$$|P_n'(t)| = \left|\sum_{c=1}^{n} q^{2c(s-2)} \sin(q^c x) \sum_{d=1}^{c} q^{-d(s-2)} \sin(q^{c-d}x)(q^{2c}-q^{2(c-d)}) \sin\left[(q^{2c}-q^{2(c-d)})t\right]\right| \le$$

$$\sum_{c=1}^{n} q^{2c(s-2)} \sum_{d=1}^{c} q^{-d(s-2)} \left(q^{2c}-q^{2(c-d)}\right) = \sum_{c=1}^{n} q^{2c(s-2+1)} \sum_{d=1}^{c} q^{-d(s-2)}(1-q^{-2d}) =$$

$$\frac{q^{2-s}}{q^{2-s}-1}\left[q^s \frac{q^{ns}-1}{q^s-1} - q^{2(s-1)} \frac{q^{2n(s-1)}-1}{q^{2(s-1)}-1}\right] - \frac{q^{-s}}{q^{-s}-1}\left[q^{s-2} \frac{q^{n(s-2)}-1}{q^{s-2}-1} - q^{2(s-1)} \frac{q^{2n(s-1)}-1}{q^{2(s-1)}-1}\right]. \quad (66)$$

Therefore, for n large enough,

$$|P_n'(t)| \le c_1\, q^{n\max\{s,2(s-1),s-2\}} = c\, q^{ns}. \quad (67)$$

Let $\delta = q^{-\alpha n}$. Then $q^n = \delta^{-1/\alpha}$ and

$$\mathrm{osc}_\delta(t) P_n \le 2c_1\, \delta q^{ns} = 2c_1\, \delta^{1-s/\alpha}. \quad (68)$$

Let

$$R_n(t) := \widetilde{P}(t) - P_n(t). \quad (69)$$

Then

$$\mathrm{osc}_\delta(t) R_n \le 2 \sum_{c=n+1}^{\infty} q^{2c(s-2)} \sum_{d=1}^{c} q^{-d(s-2)}$$

$$\le \frac{2q^{2-s}}{q^{2-s}-1} \sum_{c=n+1}^{\infty} q^{2c(s-2)+c(2-s)} = c_2\, \delta^{(s-2)/\alpha}. \quad (70)$$

To obtain a consistent estimate we must set

$$1 - \frac{s}{\alpha} = \frac{2}{\alpha} - \frac{s}{\alpha}, \quad (71)$$

which gives $\alpha = 2$. Thus

$$\mathrm{osc}_\delta(t) \widetilde{P} \le (2c_1 + c_2)\, \delta^{1-s/2}. \quad (72)$$

(b) Now we want to show that

$$W = \int_a^b \mathrm{d}t\, \left|\widetilde{P}(t+\delta) - \widetilde{P}(t-\delta)\right| \ge c\, \delta^{1-s/2}. \quad (73)$$

Set $a = 0, b = 2\pi$ for convenience. Then

$W =$

$$\int_0^{2\pi} \mathrm{d}t \left|\sum_{c=1}^{\infty} q^{2c(s-2)} \sin(q^c x) \sum_{d=1}^{c} q^{-d(s-2)} \sin(q^{c-d}x) \left\{\cos[(q^{2c}-q^{2(c-d)})(t+\delta)] - \cos[(q^{2c}-q^{2(c-d)})(t-\delta)]\right\}\right| =$$

$$\int_0^{2\pi} \mathrm{d}t \left|\sum_{c=1}^{\infty} q^{2c(s-2)} \sin(q^c x) \sum_{d=1}^{c} q^{-d(s-2)} \sin(q^{c-d}x) \left\{-2\sin[(q^{2c}-q^{2(c-d)})t] \sin[(q^{2c}-q^{2(c-d)})\delta]\right\}\right|. \quad (74)$$

Using our standard arguments, we multiply the integrand by a suitable function smaller or equal to 1,

$$W \ge \int_0^{2\pi} \mathrm{d}t\ |h(t)|\ \left|\widetilde{P}(t+\delta) - \widetilde{P}(t-\delta)\right| \ge$$

$$\left|\int_0^{2\pi} \mathrm{d}t\ h(t)\left[\widetilde{P}(t+\delta) - \widetilde{P}(t-\delta)\right]\right|. \quad (75)$$

We choose $h(t) = \sin[(q^{2c}-q^{2(c-d)})t]$ and set $\delta = q^{-2N}$. It follows that

$$W \ge 2\pi q^{(2c-d)(s-2)} \times \left|\sin(q^c x) \sin(q^{c-d}x) \sin\left(q^{2c}-q^{2(c-d)}\right)\, q^{-2N}\right|. \quad (76)$$

We now want to show that for almost all x

$$W \ge c_3\, \delta^{1-s/2} = c_3\, q^{N(s-2)}. \quad (77)$$

Set $2c-d = N$. Then

$$W \ge 2\pi\, q^{N(s-2)} \times \left|\sin(q^c x) \sin(q^{N-c}x) \sin\left(q^{2(c-N)} - q^{-2c}\right)\right|. \quad (78)$$

Thus it is enough to bound

$$\left|\sin(q^c x) \sin(q^{N-c}x) \sin[q^{2(c-N)} - q^{-2c}]\right| \quad (79)$$

from below.

Choose rational x/π. All the rational numbers in a given basis q have finite or periodic expansion. In the first case ($x/\pi = k/q^l$), we cannot find the lower bound on (79). We cannot succeed, because at these points the function $P_x(t)$ is smooth (cf. the proof of Theorem 3.4).

The other case means that x/π can be written as

$$\frac{x}{\pi} = 0.a_1 a_2 \ldots a_K \left(a_{K+1} \ldots a_{K+T}\right), \quad (80)$$

where again $(a_{K+1} \dots a_{K+T})$ denotes the periodic part. Therefore, for every $N{>}K$, $q^n x \mod \pi$ can take only one of T values: $q^{K+1}x \mod \pi, \dots, q^{K+T}x \mod \pi$. Let us take $c = 1$, $N{>}K$. Then $|\sin(q^c x)| = |\sin(qx)| > 0$ and is a constant. Note that $|\sin(q^{N-1}x)|$ takes one of T values, none of which is 0, therefore it is always bounded from below by

$$\inf_{l=1,2,\dots,T} |\sin(q^{K+l}x)| > 0. \tag{81}$$

Also, the last term can be bounded

$$\left|\sin\left(q^{-2(N-c)}{-}q^{-2c}\right)\right| \geq \sin\left(q^{-2}{-}q^{-2(N-1)}\right) \geq \sin(q^{-3}) \tag{82}$$

for $N \geq 3$. Thus for rational x/π with periodic expansion in q

$$W \geq 2\pi\, c_3\, q^{N(s-2)}, \tag{83}$$

where $c_3{=}|\sin(qx)\sin(\frac{x}{q^3})|\inf_{l=1,2,\dots,T}|\sin(q^{K+l}x)|$.

Consider now irrational x/π. Inequality (78) for $c = N$ takes the form

$$W \geq 2\pi q^{N(s-2)} \left|\sin(q^N x)\sin(x)\sin\left(1{-}q^{-2N}\right)\right| \geq c\, q^{N(s-2)} \left|\sin(q^N x)\right|, \tag{84}$$

for $N \geq 2$. Instead of showing it can be bounded from below, we will use it to prove that for almost every x

$$\dim_B \text{graph}\, P_x(t) \geq 1 + s/2. \tag{85}$$

Let

$$x_n := q^n\ (x/\pi) \mod 1. \tag{86}$$

Let

$$F_N^\alpha := \left\{x : \exists n{\geq}N \left(x_n{\leq}\tfrac{1}{q^{N\alpha}}\right) \vee \left(1{-}x_n{\leq}\tfrac{1}{q^{N\alpha}}\right)\right\}, \tag{87}$$

where $\alpha \in [0, 1]$. Let

$$F_\infty^\alpha := \bigcap_{N=1}^{\infty} F_N^\alpha. \tag{88}$$

Clearly,

$$F_N^\alpha \supset F_{N+1}^\alpha \supset F_{N+2}^\alpha \cdots \tag{89}$$

Since the Renyi map (86) preserves the Lebesgue measure, we have

$$\mu(F_N^\alpha) \leq 2\left(\frac{1}{q^{N\alpha}} + \frac{1}{q^{(N+1)\alpha}} + \frac{1}{q^{(N+2)\alpha}} + \dots\right) = \frac{2q}{q-1}\frac{1}{q^{N\alpha}}. \tag{90}$$

Therefore

$$0 \leq \mu(F_\infty^\alpha) \leq \inf_N \mu(F_N^\alpha) = 0. \tag{91}$$

It follows that for almost every $\{x_n\}$

$$\lim_{n\to\infty} \frac{\ln|\sin x_n|}{n} \geq \lim_{n\to\infty} \frac{\ln(q^{-n\alpha})}{n} \geq \lim_{n\to\infty} \frac{q^{-n\alpha}}{2n} \geq -\alpha \ln(q). \tag{92}$$

Thus for every $\alpha{>}0$ and for almost every $x/\pi \in [0, 1]$ we have

$$\dim_B \text{graph} P_x(t) = \lim \left[2 - \frac{\ln\left(\text{Var}_\delta P_x(t)\right)}{\ln(q^{-2N})}\right] \geq 2 + \lim \frac{\ln\left(\text{Var}_\delta P_x(t)\right)}{2N\ln(q)}. \tag{93}$$

But

$$\text{Var}_\delta P_x(t) \geq W, \tag{94}$$

therefore from (84)

$$\dim_B \text{graph} P_x(t) \geq 2 + \lim \frac{\ln(c){+}N(s{-}2)\ln(q){+}\ln\left|\sin(x_n\pi)\right|}{2N\ln(q)} = 1 + \frac{s}{2} + \lim \frac{\ln|\sin(x_n\pi)|}{2N\ln(q)} \geq 1 + \frac{s}{2} - \alpha. \tag{95}$$

But α is arbitrary, thus

$$\dim_B \text{graph} P_x(t) \geq 1 + s/2. \tag{96}$$

4. *For a discrete, dense set of points x_d, $D_t(x_d) = \dim_B \text{graph} P_{x_d}(t) = 1$.*

Let $x_{k,m} = \frac{m\pi}{q^k}$, where $k \in \mathbb{N}$, $m = 0, 1, \dots, q^k{-}1$. The set $\{x_{k,m}\}$ is dense in $[0, 1]$. At these points, $\Psi(x_{k,m}, t)$ is a sum of a finite number of terms

$$\Psi\left(\frac{m\pi}{q^k}, t\right) = \sqrt{\frac{2(1{-}q^{-2(2-s)})}{\pi}} \sum_{n=0}^{k-1} q^{(s-2)n} \times \sin(q^{n-k}m\pi)\, \mathrm{e}^{-\mathrm{i}\, q^{2n}t}. \tag{97}$$

Therefore,

$$\dim_B \text{graph} \left|\Psi\left(\frac{m\pi}{q^k}, t\right)\right|^2 = 1. \tag{98}$$

5. *For even q the average velocity $\frac{\mathrm{d}\langle x(t)\rangle}{\mathrm{d}t}$ is fractal with the dimension of its graph equal to $D_v = \max\{(1 + s)/2, 1\}$.*

Heuristically, this is rather obvious, because

$$\frac{\mathrm{d}\langle x(t)\rangle}{\mathrm{d}t} \approx \sum\nolimits_{k=1}^{\infty} \frac{q^{k(s-1)}}{q^{2k}} \sin(q^{2k}t) = \sum\nolimits_{k=1}^{\infty} q^{2k(s-3)/2} \sin(q^{2k}t). \tag{99}$$

Thus the average velocity is essentially a Weierstrass-like function and the dimension of its graph should be

$$2 - (3 - s)/2 = (1 + s)/2. \tag{100}$$

It is enough to consider

$$W(t) := \sum\nolimits_{k=1}^{\infty} \frac{q^{k(s-1)}}{q^{2k} - 1} \sin[(q^{2k} - 1)t]. \tag{101}$$

(a) Let

$$W_n(t) := \sum\nolimits_{k=1}^{n} \frac{q^{k(s-1)}}{q^{2k} - 1} \sin\left[(q^{2k}{-}1)t\right]. \tag{102}$$

Set $\delta = q^{-\alpha n}$. Then

$$|W_n'(t)| = \left|\sum_{k=1}^{n} q^{k(s-1)} \cos\left[(q^{2k}-1)t\right]\right|$$

$$\leq \sum_{k=1}^{n} q^{k(s-1)} \leq c_1\, \delta^{(1-s)/\alpha}, \tag{103}$$

where

$$c_1 = \frac{q^{s-1}}{q^{s-1}-1}. \tag{104}$$

Therefore,

$$\mathrm{osc}_\delta(t) W_n \leq 2c_1\, \delta^{(1-s)/\alpha}\delta = 2c_1\, \delta^{1+(1-s)/\alpha}. \tag{105}$$

Now, for

$$P_n(t) := W(t) - W_n(t), \tag{106}$$

we have

$$|P_n(t)| = \left|\sum_{k=n+1}^{\infty} \frac{q^{k(s-1)}}{q^{2k}-1} \sin\left[(q^{2k}-1)t\right]\right|$$

$$\leq \sum_{k=n+1}^{\infty} \frac{q^{k(s-1)}}{q^{2k}-1} \leq \sum_{k=n+1}^{\infty} \frac{2q^{k(s-1)}}{q^{2k}} = c_2\, \delta^{-(s-3)/\alpha}, \tag{107}$$

where

$$c_2 = \frac{2q^{s-3}}{1-q^{s-3}}. \tag{108}$$

Thus

$$\mathrm{osc}_\delta(t) P_n \leq 2c_2\, \delta^{(3-s)/\alpha}. \tag{109}$$

To obtain consistent estimates for both P_n and W_n we must set

$$1 + (1-s)/\alpha = (3-s)/\alpha, \tag{110}$$

thus $\alpha = 2$ and $\delta = q^{-2N}$. Therefore,

$$\mathrm{osc}_\delta(t)\, W \leq \mathrm{osc}_\delta(t) W_n + \mathrm{osc}_\delta(t) P_n$$

$$\leq 2(c_1+c_2)\, \delta^{2-(s+1)/2}. \tag{111}$$

(b) Consider

$$\int_a^b \mathrm{d}t\ |W(t+\delta) - W(t-\delta)| =$$

$$\int_a^b \mathrm{d}t \left|\sum_{k=1}^{\infty} \frac{q^{k(s-1)} \cos\left[(q^{2k}-1)t\right]\, \sin[(q^{2k}-1)\delta]}{q^{2k}-1}\right|$$

$$\geq \left|\int_a^b \mathrm{d}t\ h(t) f_N(t)\right| - \sum_{k\neq N} \left|\int_a^b \mathrm{d}t\ h(t) f_k(t)\right|, \tag{112}$$

where

$$f_k(t) = \frac{q^{k(s-1)}}{q^{2k}-1} \cos\left[(q^{2k}-1)t\right]\, \sin\left[(q^{2k}-1)\delta\right]. \tag{113}$$

Let $h(t) = \cos[(q^{2N}-1)t]$, $\delta = q^{-2N}$. Then

$$\left|\int_a^b \mathrm{d}t\ h(t) f_N(t)\right| = \frac{q^{N(s-1)}}{q^{2N}-1} \sin\left(1 - q^{-2N}\right) \int_a^b \mathrm{d}t\ \cos^2[(q^{2N}-1)t] \geq \frac{q^{N(s-1)}}{q^{2N}} \sin\left(\frac{\pi}{6}\right) \int_a^b \mathrm{d}t\ \cos^2[(q^{2N}-1)t]$$

$$\geq \frac{1}{2} q^{N(s-3)} \left[\frac{b-a}{2} + \frac{\sin[2b(q^{2N}-1)] - \sin[2a(q^{2N}-1)]}{4(q^{2N}-1)}\right] \geq \frac{1}{2}\delta^{(3-s)/2} \left[\frac{b-a}{2} - \frac{2\cdot 2}{4q^{2N}}\right] =$$

$$\frac{1}{2}\, \delta^{(3-s)/2} \left[\frac{1}{2}(b-a) - \delta\right] \geq \frac{1}{8}\, \delta^{(3-s)/2}(b-a). \tag{114}$$

On the other hand,

$$\left|\int_a^b \mathrm{d}t\ h(t) f_k(t)\right| = \frac{q^{k(s-1)}}{q^{2k}-1} \sin\left[(q^{2k}-1)q^{-2N}\right] \int_a^b \mathrm{d}t\ \cos[(q^{2k}-1)t] \cos[(q^{2N}-1)t]$$

$$\leq \frac{q^{k(s-1)}}{q^{2k}-1} \left|\frac{\sin[b(q^{2N}-q^{2k})] - \sin[a(q^{2N}-q^{2k})]}{2(q^{2N}-q^{2k})} + \frac{\sin[b(q^{2N}+q^{2k})] - \sin[a(q^{2N}+q^{2k})]}{2(q^{2N}+q^{2k})}\right|$$

$$\leq 2q^{k(s-3)} \left[\frac{1}{|q^{2N}-q^{2k}|} + \frac{1}{q^{2N}+q^{2k}}\right] \leq 2q^{k(s-3)} \left[\frac{1}{q^{2N}-q^{2(N-1)}} + \frac{1}{q^{2N}}\right] \leq 5q^{k(s-3)}\delta. \tag{115}$$

Therefore,

$$W \geq \frac{1}{8}\delta^{(3-s)/2}(b-a) - \sum_k 5q^{k(s-3)}\,\delta \geq \frac{1}{8}\delta^{(3-s)/2}(b-a) - \frac{5q^{s-3}}{q^{s-3}-1}\,\delta. \tag{116}$$

But $\frac{1}{2}(3-s) < 1$, thus for large enough N (small enough δ) the first term dominates the other, therefore

$$W \geq c\,\delta^{2-(1+s)/2}, \tag{117}$$

with $c = (b-a)/16$, for example.

From Theorem 1 it follows that

$$D_v = \frac{1+s}{2}. \tag{118}$$

6. *The surface $P(x,t)$ has dimension $D_{xy} = 2+\frac{1}{2}s$.*

Setting x or t constant, we have shown that oscillations are bounded by $c\delta^H$, where exponent H is one of $1, s, s/2$. We also showed the lower bound of variation is always $c\,\delta^H$, again with H being one of $1, s, s/2$. What is more, there is a dense set of points x, for which $\mathrm{Var}_\delta P_x(t) \geq c\,\delta^{s/2}$. One can take, for instance, all rational x/π with periodic q-expansion. Thus from Theorem 2 we have

$$D_{xt} = 1 + \max\{D_x, D_t\} = 2 + \frac{s}{2}. \tag{119}$$

Proof. □

5. Conclusions

In this article, we proved a theorem announced in [32] that a simple textbook problem of quantum theory — the Schröedinger equation describing a point particle in an infinite potential wall — admits continuous but nowhere differentiable solutions with fractal structure. The proposed solutions $\Psi = \Psi(x,t)$ display properties of a fractal quantum carpet, i.e., the probability density, $P(x,t) = |\Psi(t,x)|^2$, forms a fractal surface and its dimension D_{xy} is determined by the fractal dimension D_x of the cross-section $P_t(x)$.

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Appendix: Auxiliary calculations

A1. Probability density

Take the fractal wave function (36),

$$\Psi(x,t) = \frac{\sqrt{2(1-q^{2(s-2)})}}{\sqrt{\pi}} \sum_{n=0}^{\infty} q^{n(s-2)} \sin(q^n x)\, \mathrm{e}^{-\mathrm{i}\, q^{2n} t}. \tag{120}$$

Let us calculate two useful forms of the probability density $P(x,t)$

$$P(x,t) = |\Psi(x,t)|^2 = \frac{2(1-q^{2(s-2)})}{\pi} \sum_{m,n=0}^{\infty} q^{(m+n)(s-2)} \times \sin(q^n x) \sin(q^m x)\, \mathrm{e}^{-\mathrm{i}\,(q^{2n}-q^{2m})t}. \tag{121}$$

Taking $k = m+n$, $l = n$ we obtain

$$P(x,t) = \tfrac{2}{\pi}\left(1-q^{2(s-2)}\right) \sum_{k=0}^{\infty} q^{k(s-2)} \times \sum_{l=0}^{k} \sin(q^l x) \sin(q^{k-l} x)\, \mathrm{e}^{-\mathrm{i}\,(q^{2l}-q^{2(k-l)})t} = \tfrac{2}{\pi}\left(1-q^{2(s-2)}\right) \sum_{k=0}^{\infty} q^{k(s-2)} \times \sum_{l=0}^{k} \sin(q^l x) \sin(q^{k-l} x) \cos[(q^{2l}-q^{2(k-l)})t]. \tag{122}$$

Substitute $c = m$, $d = m-n$ to arrive at

$$P(x,t) = \frac{2\left(1-q^{2(s-2)}\right)}{\pi} \sum_{m=0}^{\infty} \left\{ q^{2m(s-2)} \sin^2(q^m x) + 2\sum_{n<m} q^{(m+n)(s-2)} \sin(q^n x) \sin(q^m x) \cos[(q^{2m}-q^{2n})t] \right\} =$$
$$\frac{2\left(1-q^{2(s-2)}\right)}{\pi} \left\{ \sum_{m=0}^{\infty} q^{2m(s-2)} \sin^2(q^m x) + 2\sum_{c=1}^{\infty}\sum_{d=1}^{c} q^{(2c-d)(s-2)} \sin(q^c x) \sin(q^{c-d} x) \cos[(q^{2c} - q^{2(c-d)})t] \right\} =$$
$$\frac{2\left(1-q^{2(s-2)}\right)}{\pi} \left\{ \sum_{m=0}^{\infty} q^{2m(s-2)} \sin^2(q^m x) + 2\sum_{c=1}^{\infty} q^{2c(s-2)} \sin(q^c x) \sum_{d=1}^{c} q^{-d(s-2)} \sin(q^{c-d} x) \right.$$
$$\left. \times \cos\left[(q^2-1)\, q^{2(c-d)} \sum_{a=0}^{d-1} q^{2a} t \right] \right\} = \frac{2\left(1-q^{2(s-2)}\right)}{\pi} \sum_{m=0}^{\infty} q^{2m(s-2)} \sin^2(q^m x) + \frac{4\left(1-q^{2(s-2)}\right)}{\pi} \sum_{c=1}^{\infty} q^{2c(s-2)} \sin(q^c x)$$
$$\times \sum_{d=1}^{c} q^{-d(s-2)} \sin(q^{c-d} x) \cos[(q^2-1)(q^{2(c-1)} + \cdots + q^{2(c-d)})t] =: P_x(x) + P_{xt}(x,t). \tag{123}$$

Note that the time-independent part $P_x(x) =$

$$\frac{2\left(1-q^{2(s-2)}\right)}{\pi}\sum_{m=0}^{\infty}\frac{q^{2m(s-2)}(1-\cos(q^m 2x))}{2} =$$

$$\frac{1}{\pi}-\frac{\left(1-q^{2(s-2)}\right)}{\pi}\sum_{m=0}^{\infty}q^{m(2s-4)}\cos(q^m 2x),$$

$$\frac{1}{\pi}-\frac{\left(1-q^{2(s-2)}\right)}{\pi}\sum_{m=0}^{\infty}q^{m(2s-4)}\cos(q^m 2x), \qquad (124)$$

is a Weierstrass-like function with the dimension $s' = \max\{2s-2, 1\} \in [1,2)$ (i.e., for $s \in [1, 3/2]$, $s' = 1$). From the equation (123) one immediately gets the spectrum of $P(x,t)$ — all the frequencies governing the time evolution are

$$\omega_{c,d} = (q^2-1)\left(q^{2(c-1)} + \cdots + q^{2(c-d)}\right), \qquad (125)$$

where $c = 1, 2, \ldots,$ $d = 1, 2, \ldots, c$. Thus all the frequencies divide by $q^2 - 1$ which is also the smallest frequency, so the fundamental period of $P(x,t)$ is $2\pi/(q^2-1)$.

A2. Average velocity

Let us study the behavior of $\langle x\rangle$.

$$\langle x\rangle = \int_0^{\pi} \mathrm{d}x\; x\,|\Psi|^2 = \frac{\pi}{2} - \frac{16}{\pi}\left(1-q^{2(s-2)}\right)$$

$$\times \sum\nolimits_{k=1}^{\infty}\frac{q^{k(s-1)}}{(q^{2k}-1)^2}\cos[(q^{2k}-1)t]. \qquad (126)$$

The above expression is valid only for even q. For odd q we have just the first term, which is $\pi/2$.

The average $x(t)$ is of class $\mathcal{C}^1$, because its derivative is given by an absolutely convergent series

$$\left|\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t}\right| = \left|\frac{16\left(1-q^{2(s-2)}\right)}{\pi}\sum_{k=1}^{\infty}\frac{q^{k(s-1)}\sin[(q^{2k}-1)t]}{q^{2k}-1}\right|$$

$$\leq 2c\sum_{k=1}^{\infty}\frac{q^{k(s-1)}}{q^{2k}} = 2c\,\frac{q^{s-3}}{1-q^{s-3}}. \qquad (127)$$

In Sect. 4 we show that (127) is fractal, while for odd q the average velocity $|\mathrm{d}\langle x\rangle/\mathrm{d}t|$, of course, is not. This seemingly strange behavior is caused by the fact that

$$\int_0^{\pi} \mathrm{d}x\;\sin(nx)\sin(mx) \qquad (128)$$

is non-zero only for m, n of different parity. However, if one slightly disturbs our function, for instance, by changing an arbitrary number of terms to the next higher or lower eigenstates, the dimensions D_x and D_t will not be altered, but the average velocity will become fractal. In other words, with probability one, independently of the parity of q, the average velocity of the wave function

$$\Phi_0(x,t) = M_0\sum_{n=1}^{\infty}q^{n(s-2)}\sin\left[(q^n \pm 1)x\right]$$

$$\times\,\mathrm{e}^{-\mathrm{i}\,(q^n\pm 1)^2 t}. \qquad (129)$$

is fractal characterized by the same dimensions D_x and D_t as the function currently studied.

An explicit example of a similar function for odd q it is

$$\Phi_1(x,t) = M_1\Big[2^{s-2}\sin(2x)\,\mathrm{e}^{-\mathrm{i}\,2^2 t}$$

$$+\sum_{n=1}^{\infty}q^{n(s-2)}\sin(q^n x)\,\mathrm{e}^{-\mathrm{i}\,q^{2n}t}\Big]. \qquad (130)$$

One can see the only difference between this example and the original one (36) is in the *first* term. This difference accounts for the smoothness or roughness of the average velocity. It is very interesting because normally one expects that it is the *asymptotic* behavior that determines the fractal dimension. Here we have an exactly opposite case: a change in the first term (varying most slowly) of a series changes the dimension of a complicated function $\langle v\rangle$.

The average velocity of the wave packet (130) is smooth for even q and fractal for odd q. A function, which gives fractal average velocity for both even and odd q, is

$$\Phi_2(x,t) = M_2\Big[2^{s-2}\sin(2x)\,\mathrm{e}^{-\mathrm{i}\,2^2 t}$$

$$+\sum_{n=0}^{\infty}q^{n(s-2)}\sin(q^n x)\,\mathrm{e}^{-\mathrm{i}\,q^{2n}t}\Big] =$$

$$M_2\left[2^{s-2}\sin(2x)\,\mathrm{e}^{-\mathrm{i}\,2^2 t} + \frac{1}{N}\,\Psi(x,t)\right], \qquad (131)$$

where M_2 is the normalization constant. On the other hand,

$$\Phi_3(x,t) = M_3\sum_{n=1}^{\infty}q^{n(s-2)}\sin(q^n x)\,\mathrm{e}^{-\mathrm{i}\,q^{2n}t} \qquad (132)$$

gives smooth average velocity for both even and odd q.

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